GEOMETRICALLY SIMPLE LOGARITHMIC WEIR

By K. Keshava Murthy,[1] H. S. Ramesh,[2] and M. N. Shesha Prakash[3]

ABSTRACT: This paper discusses the design and experimental verification of a geometrically simple logarithmic weir. The weir consists of an inward trapezoidal weir of slope 1 horizontal to n vertical, or 1 in n, over two sectors of a circle of radius R and depth d, separated by a distance $2t$. The weir parameters are optimized using a numerical optimization algorithm. The discharge through this weir is proportional to the logarithm of head measured above a fixed reference plane for all heads in the range $0.23R \leq h \leq 3.65R$ within a maximum deviation of $\pm 2\%$ from the theoretical discharge. Experiments with two weirs show excellent agreement with the theory by giving a constant average coefficient of discharge of 0.62. The application of this weir to the field of irrigation, environmental, and chemical engineering is highlighted.

INTRODUCTION AND BACKGROUND

Sensitivity is a very important characteristic of a flow-measuring device in hydrometry (Troskolanski 1960). Logarithmic weirs, which give a discharge proportional to the logarithm of the head, are a sensitive flow-measuring device, because the error caused in the discharge for the same error committed in the head is less than in linear, exponential, and conventional weirs.

It is clear from the works of Cowgill (1944) and Banks (1954) that the logarithmic weirs belong to the class of "compensating weirs" (Keshava Murthy and Gopalakrishna Pillai 1978), invariably requiring a base for their design. Govinda Rao and Keshava Murthy (1966) designed logarithmic weirs with a rectangular base of width $2W$ and depth a, over which the designed complementary weir is fitted. The discharge through this weir is proportional to the logarithm of head measured above a fixed reference plane, which is unique for every weir fixed according to the slope-discharge continuity theorem (Keshava Murthy and Seshagiri 1968). This weir was improved by the universalization of its coordinates by Chandrasekaran and Lakshmana Rao (1970). However, these exact weirs have bases, and the curved profile designed by the application of Abel's integral equation (Govinda Rao and Keshava Murthy 1966) is complex and often difficult to fabricate under field conditions. In the case of exact weirs, the width of the weir becomes zero at comparatively low heights, restricting the range of measurements, and this height can be raised only by enlarging the size of the base weir. This type of enlargement is not desirable because considerable discharge passes through the base weir itself before the flow enters the proportional portion of the weir. Further, it may not always be possible to accommodate any size of the base weir in a given channel section. The present investigation is an attempt to overcome the foregoing difficulties. A geometrically simple weir in the form of an inverted V-notch was theoretically analyzed for its linear head-discharge characteristics by Keshava Murthy and Giridhar (1989). In this paper a geometrically simple weir in the form of two sectors, over which there is an inward trapezium, is analyzed with regard to its logarithmic relationship, including its range of applicability. A new numerical optimization algorithm is developed to design the parameters of the weir to maximize the range of applicability within a prefixed maximum allowable error.

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FORMULATION OF PROBLEM

Fig. 1 is a definition sketch of a geometrically simple logarithmic weir. The discharge through a symmetrical sharp-crested weir is defined by the profile $y = f(x)$ (as shown in Fig. 1) where x and y are vertical and horizontal axes, neglecting the velocity of approach and surface tension effects (as the flow measurement starts beyond a certain minimum base-flow depth), described by Weisbach as

$$q = 2C_d\sqrt{2g} \int_0^h \sqrt{h - x} f(x)\, dx \tag{1}$$

where q = discharge through the weir; h = head above the crest; g = acceleration due to gravity; and C_d = coefficient of discharge. C_d is assumed to be constant for streamlined flows through sharp-crested weirs. Similar to conventional weirs, the constant has to be ascertained through experiments, which may vary within $\pm 1\%$ of the average C_d. The coefficient of discharge depends on parameters such as the

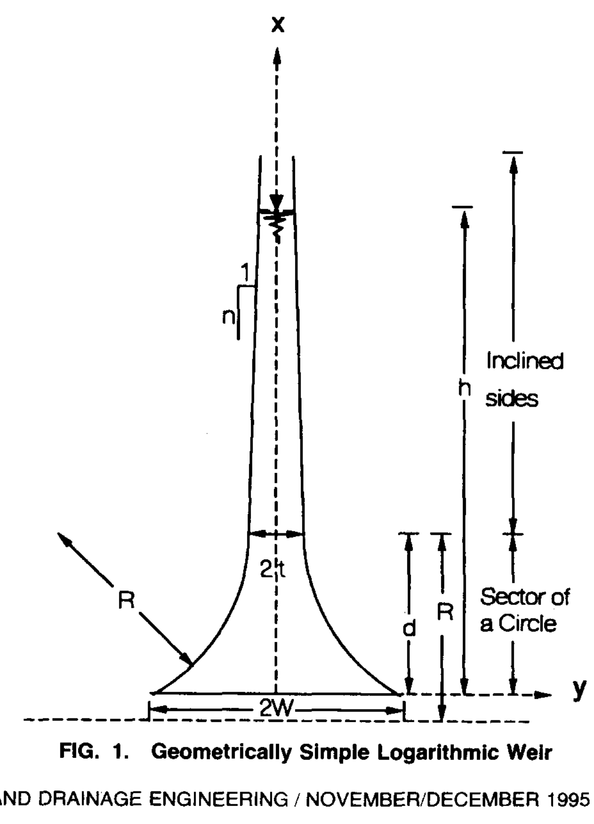

FIG. 1. Geometrically Simple Logarithmic Weir

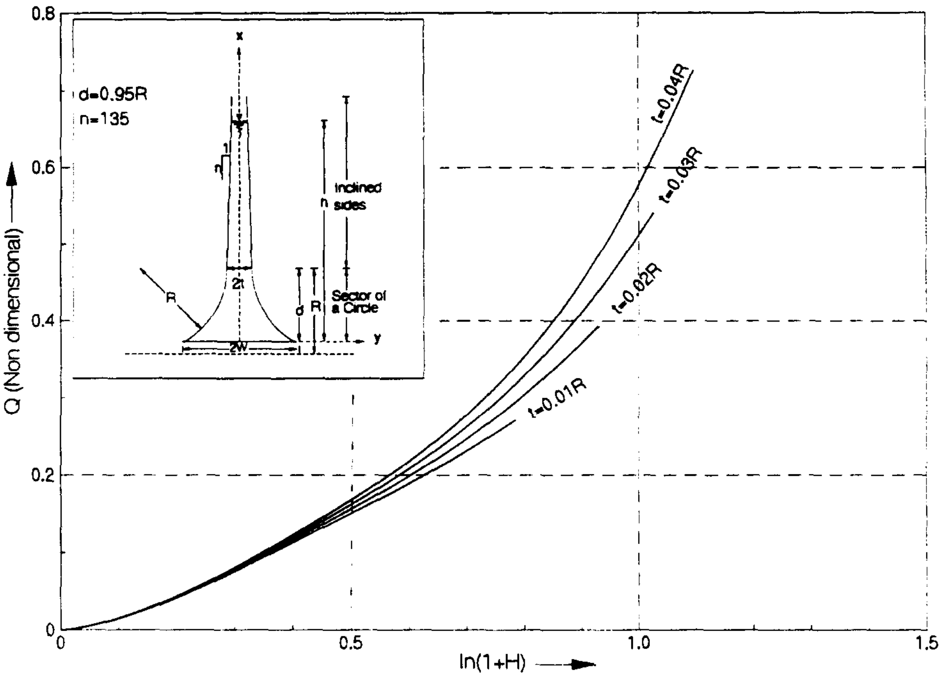

FIG. 2. Variation of Theoretical Head-Discharge Curves

ratio of the flow depth to the crest height from the bed, ratio of the width of the weir to the breadth of the channel, viscosity, and surface tension. The exact value of C_d has to be ascertained by detailed experiments, necessary for the standardization of the weir, which are beyond the scope of this paper.

For flows in the base weir region

$$f(x) = R + t - \sqrt{R^2 - (d - x)^2}$$

and the discharge is given by

$$q = 2C_d\sqrt{2g}\int_0^d \sqrt{h - x}[R + t - \sqrt{R^2 - (d - x)^2}]\,dx \quad \ldots 0 \le h \le d \tag{2a}$$

When the flow enters the inclined sides (i.e., inward trapezoidal weir), the discharge is

$$q = 2C_d\sqrt{2g}\left\{\int_0^d \sqrt{h - x}[R + t - \sqrt{R^2 - (d - x)^2}]\,dx + \frac{2}{3}\left[(h - d)^{(3/2)} + \frac{2}{5n}(h - d)^{(5/2)}\right]\right\} \quad \ldots 0 \le h \le h_{\max} \tag{2b}$$

where R = radius of the sector of the circle; d = depth of the sector of the circle; t = half top width of sectors of the circle; and n = sideslope of the inclined sides.

For the sake of convenience, (2a) and (2b) can be expressed in the nondimensional form as

$$Q = \int_0^H [1 + T - \sqrt{1 - (D - X)^2}]\sqrt{H - X}\,dX \quad \ldots 0 \le H \le D \tag{3a}$$

and

$$Q = \int_0^D \sqrt{H - X}[1 + T - \sqrt{1 - (D - X)^2}]\,dX + \frac{2}{3}\left\{(1 + T)H^{(3/2)} - \left[1 + \frac{2}{5n}(H - D)\right](H - D)^{(3/2)}\right\} \quad \ldots D \le H \le H_{\max} \tag{3b}$$

where $Q = q/KR^{(5/2)}$; $K = 2C_d\sqrt{2g}$; $H = h/R$; $X = x/R$; $T = t/R$; and $D = d/R$.

Evaluating the foregoing integrals by Simpson's one-third rule, taking a sufficiently small step size (1/1,000), we obtain a high degree of accuracy. The discharge graphs of Q versus $\ln(1 + H)$ for various values of T for a given value of n, are shown on a semilog plot in Fig. 2. The graph is almost linear for a wide range of heads, which supports the assumption of a near-logarithmic relation between Q and H over a certain range of head within a prefixed maximum error.

ANALYSIS

Let a logarithmic relationship between the head and discharge be of the nondimensional form

$$Q_{\ln} = b\ln(1 + H) + C \tag{4}$$

where b = a constant of proportionality; and C = discharge intercept, such that $Q_{\ln}$ represents the same discharge characteristics within a certain range and within a prescribed maximum percentage deviation of error, E [usually taken as ± 2 Troskolanski (1960) which has been adopted in our analysis] from the theoretical one.

Let $K_u = [1 + (E/100)]$ and $K_d = [1 - (E/100)]$, such that

$$f_1(H) = K_uQ;\quad f_2(H) = K_dQ \tag{5a,b}$$

define two explicit curves forming the upper and lower bounds for the logarithmic function to lie within, as shown in Fig. 3. The $f_1(H)$ and $f_2(H)$ curves are plotted against $H_d[H_d = \ln(1 + H)]$. Each point on the $f_1(H)$ curve (b_{ij}) is joined to every point of the $f_2(H)$ curve (a_i) successively and for each line ($\overrightarrow{a_ib_{ij}}$), the proportionality range (PR), i.e., the horizontal

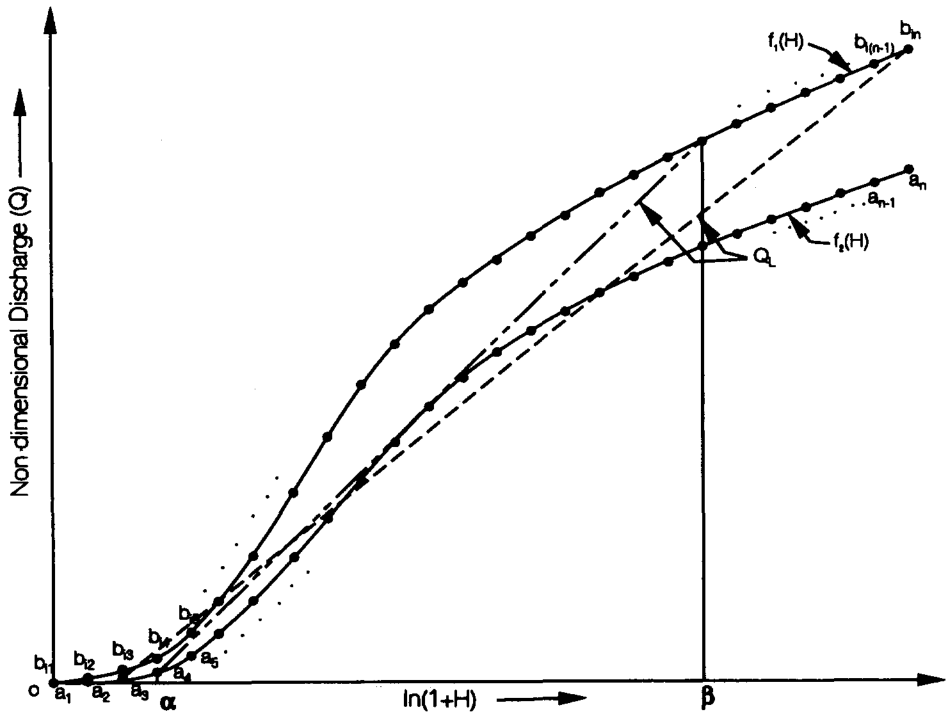

FIG. 3. Optimization Procedure to Obtain Proportionality Range (*PR*)

TABLE 1. Numerical Optimization Algorithm

Trial set number (1)	Point on lower curve $f_2(H)$ *A* (2)	Point on upper curve $f_1(H)$ *B* (3)	Slope computed [Eq. (7)] m_{ij} (4)	Intercept computed [Eq. (8)] C_{ij} (5)	*PR* PR_{ij} (6)	Maximum *PR* of trial set PR_i (7)
1	a_1	b_{1n}	m_{1n}	C_{1n}	PR_{1n}	PR_1
		$b_{1(n-1)}$	$m_{1(n-1)}$	$C_{1(n-1)}$	$PR_{1(n-1)}$	
		.	.	.	.	
		.	.	.	.	
		b_{12}	m_{12}	C_{12}	PR_{12}	
		b_{11}	m_{11}	C_{11}	PR_{11}	
2	a_2	b_{2n}	m_{2n}	C_{2n}	PR_{2n}	PR_2
		$b_{2(n-1)}$	$m_{2(n-1)}$	$C_{2(n-1)}$	$PR_{2(n-1)}$	
		.	.	.	.	
		.	.	.	.	
		b_{22}	m_{22}	C_{22}	PR_{22}	
		b_{21}	m_{21}	C_{21}	PR_{21}	
.	.	.	.	.	.	.
.	.	.	.	.	.	.
l	a_l	b_{ln}	m_{ln}	C_{ln}	PR_{ln}	PR_l
		$b_{l(n-1)}$	$m_{l(n-1)}$	$C_{l(n-1)}$	$PR_{l(n-1)}$	
		.	.	.	.	
		.	.	.	.	
		b_{ll}	m_{ll}	C_{ll}	PR_{ll}	
		b_{l1}	m_{l1}	C_{l1}	PR_{l1}	

Note: where $i \rightarrow 1, l$; $l \leq n/2$; and $j \rightarrow n, 1$.

projection of the portion of the line in the bound region of Fig. 3 on the abscissa, is calculated. The maximum proportionality range (*PR*) and its applicable limits are

$$PR_{ij} = B_{ij} - A_i \tag{6}$$

where $A_i = e^{\alpha_i} - 1$; and $B = e^{\beta_i} - 1$, α_i and β_i being the starting and ending points of the proportionality range projected on the logarithmic axis. The slope *m* and the discharge intercept *C* of the straight line are evaluated as

$$m_{ij} = \frac{f_{1ij}(H) - f_{2i}(H)}{H_{d_{ij}} - H_{d_i}} \tag{7}$$

and

$$C_{ij} = f_{1ij}(H) - m_{ij}H_{d_i} \tag{8}$$

OPTIMIZATION OF WEIR PARAMETERS

From the preceding section, the near-logarithmic relationship in the head-discharge plot is simplified to

$$f(H) = m \ln(1 + H) + C \tag{9}$$

between the error curves with the maximum horizontal projection on a semilog plot. On comparing (4) and (9) we get

$$Q_{\ln} = f(H); \quad B = m; \quad C = C \tag{10a-c}$$

To obtain the greatest projection in the region formed by the two monotonically increasing curves $f_2(Q)$ and $f_1(Q)$ shown in Fig. 3, a systematic optimization procedure was developed in "Fortran 77" and run on an IBM RISC machine. The algorithm is given in Table 1.

ESTIMATION OF WEIR PARAMETERS

Fig. 2 shows the theoretical head-discharge curve on a semilog graph for particular values of *n* and *D* for four values of *T*. For each value of *T*, there is a distinct head range in which the theoretical head-discharge relation is nearly logarithmic.

To optimize the range of the aforementioned logarithmic relationship, the proportionality range is calculated for each set of *D*, *n*, and *T* values and plotted. Fig. 4 shows the variation of proportionality range with respect to *n* for a particular value of *D* and *T*. By comparing the plots, we obtain the optimum parameters of the designed weir as $d = 0.95R$, $t = 0.02R$, and $n = 135$. The proposed logarithmic head-discharge relationship to replace the theoretical relationship is

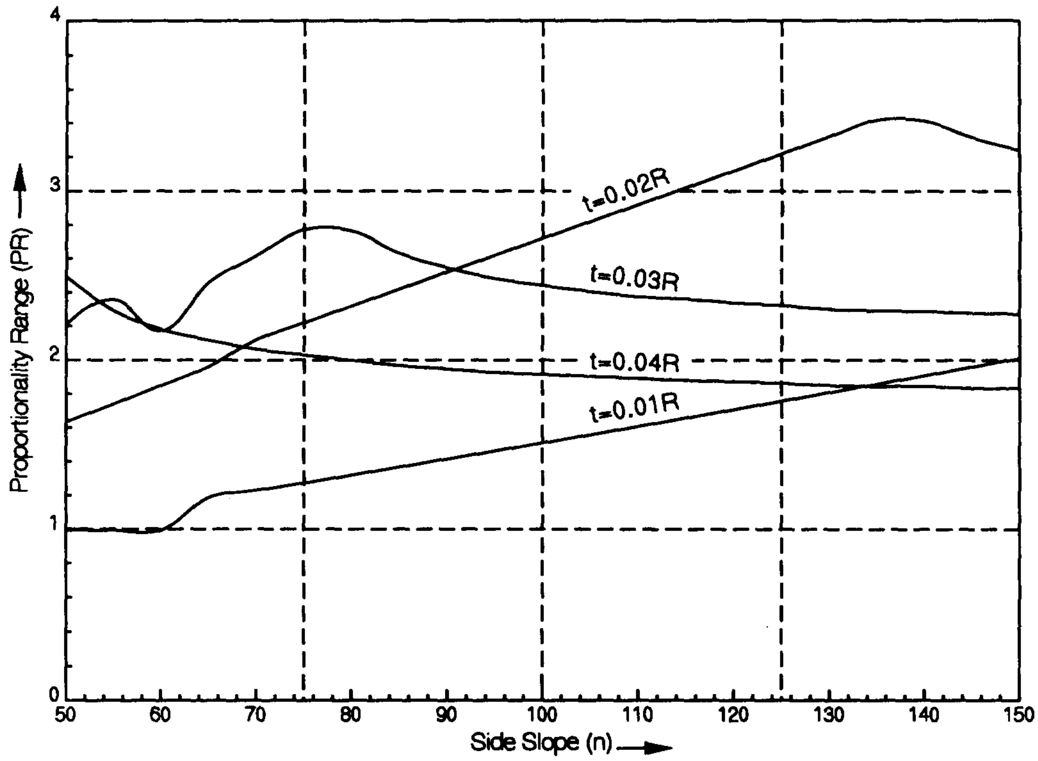

FIG. 4. Variation of Proportionality Range versus Side Slope

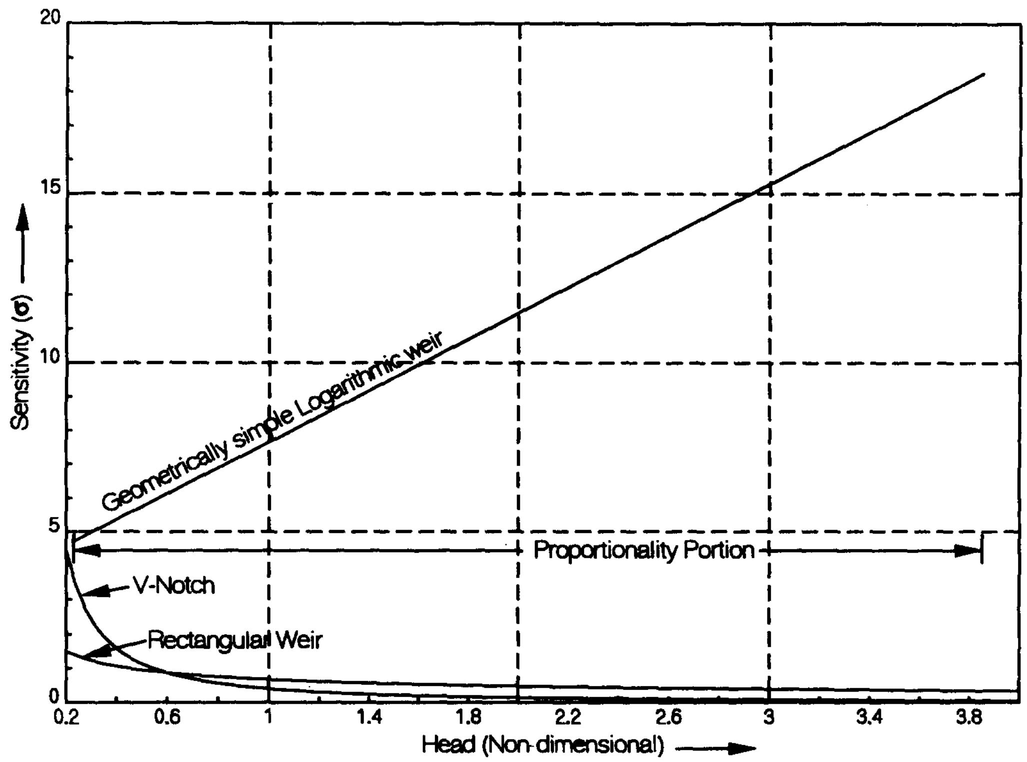

FIG. 5. Variation of Sensitivity with Head

$$Q_{ln} = 0.26186 \ln(1 + H) - 0.01521, \quad 0.23 \le H \le 3.65 \quad (11a)$$

$$Q_{ln} = 0.26186 \ln\left(0.9436 \frac{h_d}{R}\right) = 0.26186 \ln\left(\frac{h_d}{R_1}\right)$$

$$0.23R \le h \le 3.65R \quad (11b)$$

where $h_d = (R + h)$; $R_1 = 1.0598R$; and h = depth of flow measured above the crest of the weir.

Dimensionally, the proposed logarithmic head-discharge relationship can be expressed as

$$q_{ln} = 2C_d\sqrt{2g}R^{(5/2)}0.26186 \ln(h_d/R_1) \quad (11c)$$

SENSITIVITY OF WEIR

Sensitivity σ of a weir is defined (Troskolanski 1960) as the elementary increase in head over the elementary increase

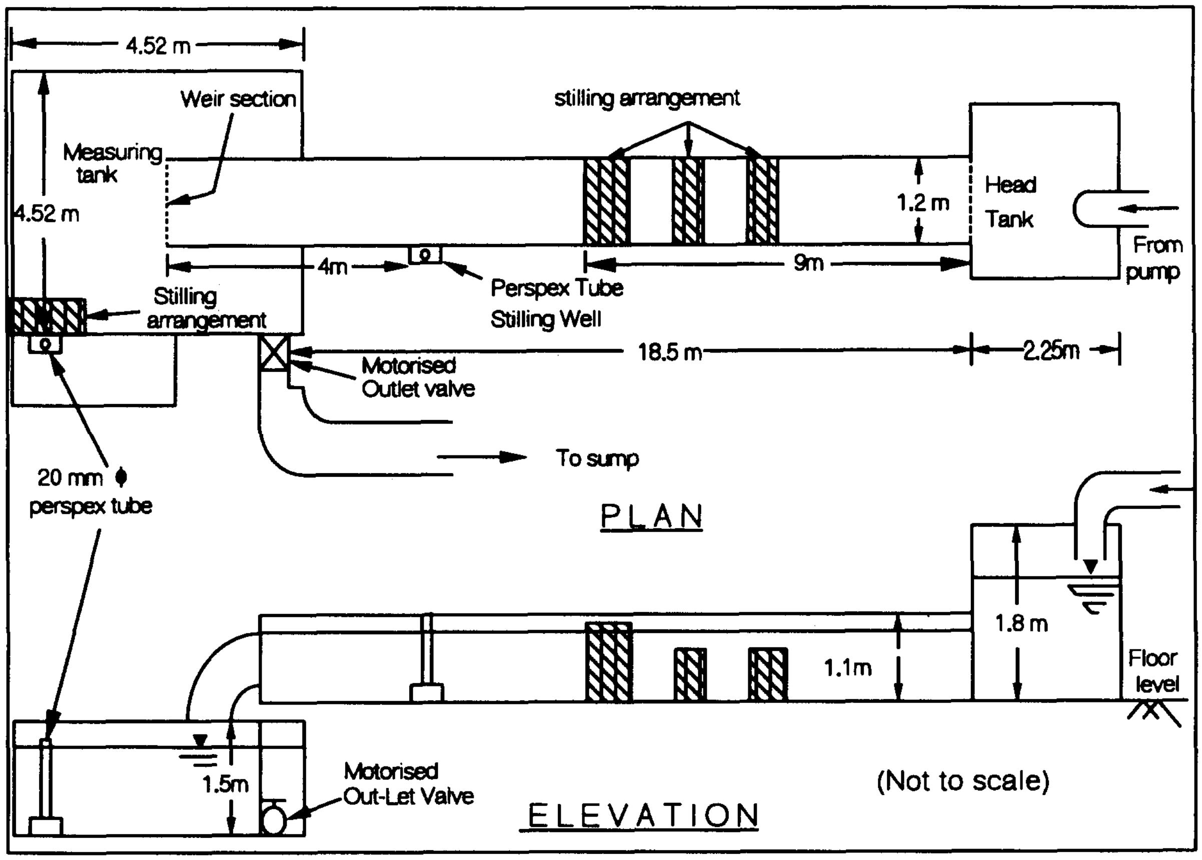

FIG. 6. Experimental Setup

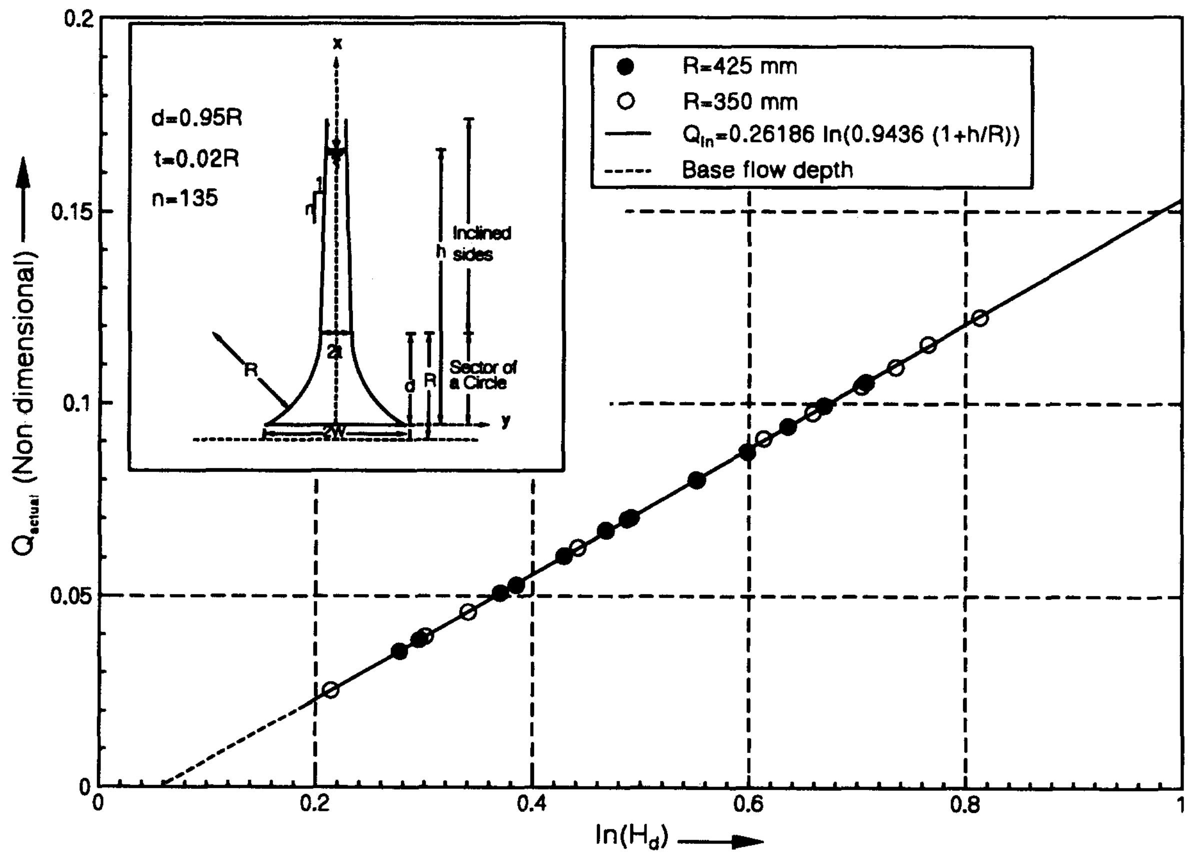

FIG. 7. Experimental Head-Discharge Plot

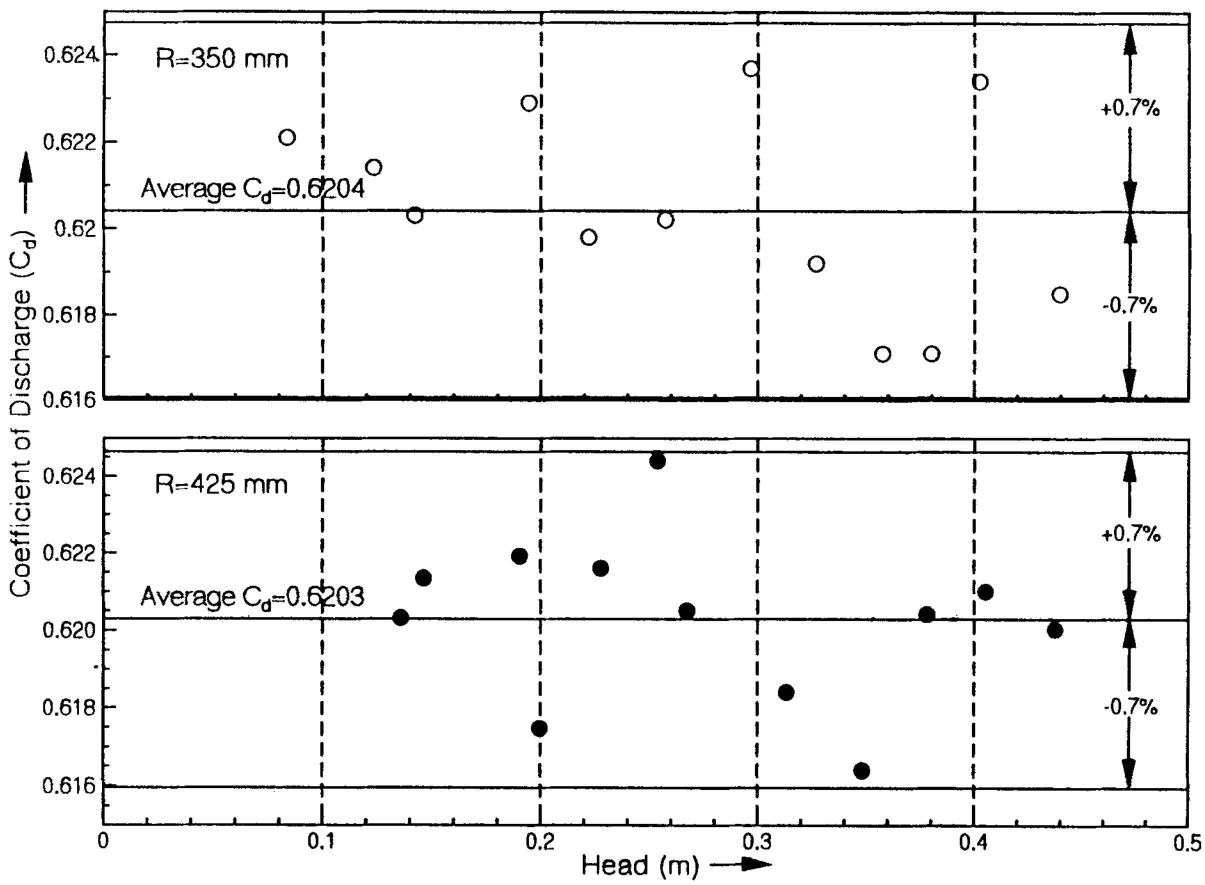

FIG. 8. Variation of C_d with Head

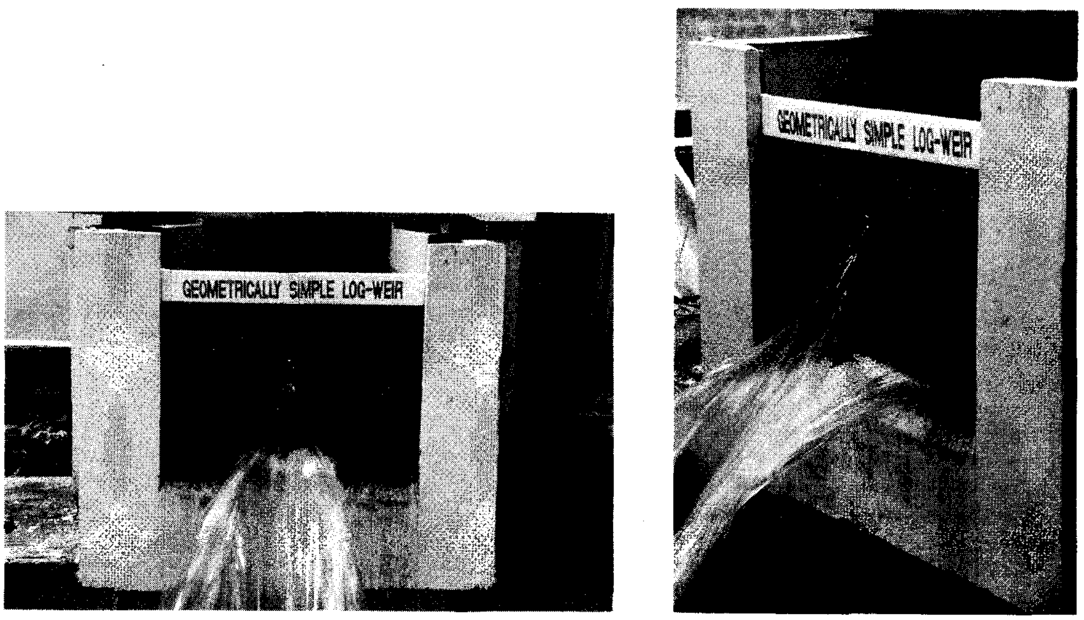

FIG. 9. Discharging Weirs: (a) Front; (b) Side

in the rate of flow. For the designed weir, the σ is obtained from (11*a*) as

$$\sigma = \frac{dH}{dQ} = \frac{(1 + H)}{0.26186} = 3.82(1 + H) \quad (12)$$

The sensitivity of the designed weir increases with the head as shown in Fig. 5.

EXPERIMENTAL VERIFICATION

The laboratory setup used to conduct experiments on the designed weirs is shown in Fig. 6. The dimensions of the weirs chosen for the experiments were R = 425 mm and R = 350 mm, respectively. The profile of the weir was cut on a nibbling machine using 914.4 mm × 609.6 mm × 6.5 mm thick mild steel (MS) sheets. The experimental weirs were fixed at the end of the channel with their crest 150 mm above the channel bed. The head above the weir was measured using an electronic point gauge with a least count of 0.01 mm, located 4 mm upstream of the weir section. The time required to collect a fixed volume of water in the measuring tank (4.52 m × 4.52 m × 1.5 m) was computed from an electronic timer triggered automatically by signals from an electronic switch attached to the level indicators.

ANALYSIS OF RESULTS

Fig. 7 shows the Q_{actual} versus ln H_d plot. Above a certain minimum depth, the plot is almost linear, which confirms the theory. The variation of the coefficient of discharge with respect to the head is plotted in Fig. 8. The coefficient of discharge for any head within the fixed range does not deviate more than ±0.7% from the average coefficient of discharge $C_d = 0.62$; this supports the assumption of a constant C_d in our analysis. The measuring capacity of the weir is indicated by the ratio of $Q_{max}/Q_{min} \approx 10$. Fig. 9 shows the front and side views of the discharging weirs.

PRACTICAL APPLICATION

For field measurements in lined irrigation canals, it is very difficult to measure the head accurately, which results in erroneous discharge computations. For the designed weir, the error in the discharge for the same error committed in the head is less than the error in linear, exponential, and conventional weirs. In addition, the simple geometrical shape of the weir and the elimination of the narrow neck portion at the crest, with the curtailment of the depth of the quadrant, makes it easy to fabricate under general field condition. Thus, the weir can be used in practice as a simple and accurate flow-measuring device in lined irrigation canals. It is also useful as an automatic flow recorder as it gives a relatively lower variation of discharge corresponding to higher variations in the head. This weir is very sensitive because of its logarithmic head-discharge relationship. Therefore, it can be used as a dosing device in automatic sampling chemical plants, where the discharge fluctuations are low and the required head variations are very high, to regulate the flow automatically through a float-regulated mechanism. A design example is given in Appendix I.

SUMMARY

It was shown that the weir formed by the two sectors of a circle of radius R and depth d separated by a distance $2t$, with an inward trapezium of sideslopes 1 in n (1 horizontal to n vertical) above it, can be used for a logarithmic head-discharge relationship within a maximum deviation of ±2% from the theoretical discharge, over a range of head determined by the values of (t/R), d, and n.

The curtailment of the quadrant depth above the crest by $5\%R$ removes sharp angles and errors in fabrications.

The weir parameters, i.e., proportionality range, base-flow depth, and datum constant vary with such dimensions of the weir as t, d, and n.

For $t = 0.02R$, $d = 0.95R$, and $n = 135$, a maximum range of proportionality was achieved. The discharge through the weir is proportional to the logarithm of the depth measured above the datum for all flows in the range $0.23R \leq h \leq 3.65R$.

The measuring capacity of the designed weir is relatively high $Q_{max}/Q_{min} \approx 10$.

The weir has a finite width within the measuring range.

Experiments with typical weirs of radii $R = 425$ mm and 350 mm show good agreement with the theory by giving a constant average coefficient of discharge $C_d = 0.62$.

With its geometrical simplicity and high sensitivity, the weir could be used in practice as a flow recorder in irrigation canals and a float-regulated dosing device in chemical plants.

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APPENDIX I. DESIGN EXAMPLE

To design a geometrically simple logarithmic weir, let it be required for an open channel of width 1 m to pass a maximum discharge of 0.25 m³/s.

Solution

$q_{ln} = 0.25$ m³/s. Assume $C_d = 0.62$. From (11*a*), the dimensional form is given by

$$q_{ln} = 2C_d\sqrt{2g}R^{5/2}0.26186 \ln\left[0.9436\left(1 + \frac{h}{R}\right)\right] \quad (13)$$

where q_{ln} is in m³/s, and h and R are in meters. The maximum measurable head is $3.65R$, which has to correspond with the maximum discharge 0.25 m³/s. Substituting for q_{ln} and C_d, we get

$$0.25 = 0.62 \times 8.8589 \times R^{5/2} \times 0.26186 \ln[0.9436(1 + 3.65)],$$

$$R = 0.4246 \text{ m} \quad (14)$$

For the sake of convenience let us assume $R = 425$ mm. The dimensions and other parameters of the geometrically simple logarithmic weir are:

1. Depth of the sectors of the circle = d = 403.75 mm.
2. Half crest width of the weir = w = 300.80 mm.
3. Distance between sectors of the circle = $2t$ = 17.00 mm.
4. The side slope of the inward trapezoidal weir = n = 135.
5. Proportionality range = PR = $3.42R$.
6. Base-flow depth = h_{min} = 97.75 mm.
7. Maximum measurable head = h_{max} = 1,551.25 mm.
8. Minimum measurable discharge = q_{min} = 0.025227 m³/s.
9. Maximum measurable discharge = q_{max} = 0.250447 m³/s.
10. $Q_{max}/Q_{min} \approx 10$.

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APPENDIX III. NOTATION

The following symbols are used in this paper:

A = base-flow depth or lower limit of proportionality range;
B = upper limit of the proportionality range;
b = proportionality constant in the logarithmic head-discharge relationship;
C = discharge intercept;
C_d = coefficient of discharge;
D = d/R;
d = depth of the sector of the circle;
E = relative error in discharge;
$f(H)$ = head-discharge function;
$f_1(H), f_2(H)$ = curves defining the permissible region for $Q_{\ln}$ to lie in Q versus $\ln(1 + H)$ plot;
g = acceleration due to gravity;
H = h/R
H_d = h_d/R;
H_{max}, H_{min} = nondimensional heads at the upper and lower limit of the logarithmic range, respectively;
h = head above the weir crest;
h_d = $(R + h)$;
h_{max} = $(t \times n)$ = maximum head in the weir;
i, j = subscripts used to indicate the position of head-discharge points;
K = $2C_d\sqrt{2g}$, a dimensional constant;
K_d = $[1 - (E/100)]$;
K_u = $[1 + (E/100)]$;
m = slope constant;
n = side slope of the inclined sides of the weir;
PR = proportionality range;
Q = $q/KR^{(5/2)}$;
$Q_{\ln}$ = proposed nondimensional logarithmic head-discharge relationship;
Q_{max}, Q_{min} = nondimensional discharges at the upper and lower limit of the logarithmic range, respectively;
q = discharge;
$q_{\ln}$ = proposed logarithmic head-discharge relationship;
R = radius of the sector of a circle;
R_1 = constant = $1.0598R$;
T = t/R;
t = half top width of the base weir (sector of a circle);
W = w/R;
w = half crest width of practical constant-accuracy linear weir;
X, Y = x/R and y/R, respectively;
x, y = vertical and horizontal coordinates, respectively;
α = lower limit of proportionality range on the logarithmic axis;
β = upper limit of the proportionality range on the logarithmic axis; and
σ = sensitivity.