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Introduction to Smooth Manifolds

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Chapter 1
Smooth Manifolds

This book is about *smooth manifolds*. In the simplest terms, these are spaces that locally look like some Euclidean space $\mathbb{R}^n$, and on which one can do calculus. The most familiar examples, aside from Euclidean spaces themselves, are smooth plane curves such as circles and parabolas, and smooth surfaces such as spheres, tori, paraboloids, ellipsoids, and hyperboloids. Higher-dimensional examples include the set of points in $\mathbb{R}^{n+1}$ at a constant distance from the origin (an *n-sphere*) and graphs of smooth maps between Euclidean spaces.

The simplest manifolds are the topological manifolds, which are topological spaces with certain properties that encode what we mean when we say that they "locally look like" $\mathbb{R}^n$. Such spaces are studied intensively by topologists.

However, many (perhaps most) important applications of manifolds involve calculus. For example, applications of manifold theory to geometry involve such properties as volume and curvature. Typically, volumes are computed by integration, and curvatures are computed by differentiation, so to extend these ideas to manifolds would require some means of making sense of integration and differentiation on a manifold. Applications to classical mechanics involve solving systems of ordinary differential equations on manifolds, and the applications to general relativity (the theory of gravitation) involve solving a system of partial differential equations.

The first requirement for transferring the ideas of calculus to manifolds is some notion of "smoothness." For the simple examples of manifolds we described above, all of which are subsets of Euclidean spaces, it is fairly easy to describe the meaning of smoothness on an intuitive level. For example, we might want to call a curve "smooth" if it has a tangent line that varies continuously from point to point, and similarly a "smooth surface" should be one that has a tangent plane that varies continuously. But for more sophisticated applications it is an undue restriction to require smooth manifolds to be subsets of some ambient Euclidean space. The ambient coordinates and the vector space structure of $\mathbb{R}^n$ are superfluous data that often have nothing to do with the problem at hand. It is a tremendous advantage to be able to work with manifolds as abstract topological spaces, without the excess baggage of such an ambient space. For example, in general relativity, spacetime is modeled as a 4-dimensional smooth manifold that carries a certain geometric structure, called a

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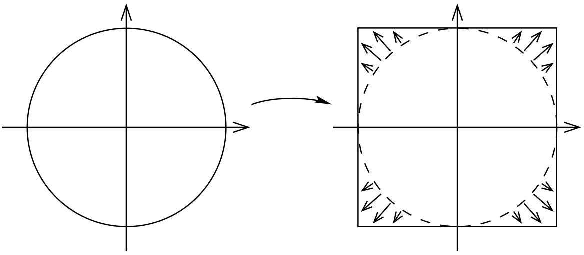

Fig. 1.1 A homeomorphism from a circle to a square

Lorentz metric, whose curvature results in gravitational phenomena. In such a model there is no physical meaning that can be assigned to any higher-dimensional ambient space in which the manifold lives, and including such a space in the model would complicate it needlessly. For such reasons, we need to think of smooth manifolds as abstract topological spaces, not necessarily as subsets of larger spaces.

It is not hard to see that there is no way to define a purely topological property that would serve as a criterion for "smoothness," because it cannot be invariant under homeomorphisms. For example, a circle and a square in the plane are homeomorphic topological spaces (Fig. 1.1), but we would probably all agree that the circle is "smooth," while the square is not. Thus, topological manifolds will not suffice for our purposes. Instead, we will think of a smooth manifold as a set with two layers of structure: first a topology, then a smooth structure.

In the first section of this chapter we describe the first of these structures. A *topological manifold* is a topological space with three special properties that express the notion of being locally like Euclidean space. These properties are shared by Euclidean spaces and by all of the familiar geometric objects that look locally like Euclidean spaces, such as curves and surfaces. We then prove some important topological properties of manifolds that we use throughout the book.

In the next section we introduce an additional structure, called a *smooth structure*, that can be added to a topological manifold to enable us to make sense of derivatives.

Following the basic definitions, we introduce a number of examples of manifolds, so you can have something concrete in mind as you read the general theory. At the end of the chapter we introduce the concept of a *smooth manifold with boundary*, an important generalization of smooth manifolds that will have numerous applications throughout the book, especially in our study of integration in Chapter 16.

Topological Manifolds

In this section we introduce topological manifolds, the most basic type of manifolds. We assume that the reader is familiar with the definition and basic properties of topological spaces, as summarized in Appendix A.

Suppose M is a topological space. We say that M is a ***topological manifold of dimension n*** or a ***topological n-manifold*** if it has the following properties:

- M is a ***Hausdorff space***: for every pair of distinct points $p, q \in M$, there are disjoint open subsets $U, V \subseteq M$ such that $p \in U$ and $q \in V$.
- M is ***second-countable***: there exists a countable basis for the topology of M.
- M is ***locally Euclidean of dimension n***: each point of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

The third property means, more specifically, that for each $p \in M$ we can find

- an open subset $U \subseteq M$ containing p,
- an open subset $\widehat{U} \subseteq \mathbb{R}^n$, and
- a homeomorphism $\varphi\colon U \to \widehat{U}$.

► **Exercise 1.1.** Show that equivalent definitions of manifolds are obtained if instead of allowing U to be homeomorphic to *any* open subset of $\mathbb{R}^n$, we require it to be homeomorphic to an open ball in $\mathbb{R}^n$, or to $\mathbb{R}^n$ itself.

If M is a topological manifold, we often abbreviate the dimension of M as $\dim M$. Informally, one sometimes writes "Let M^n be a manifold" as shorthand for "Let M be a manifold of dimension n." The superscript n is not part of the name of the manifold, and is usually not included in the notation after the first occurrence.

It is important to note that every topological manifold has, by definition, a specific, well-defined dimension. Thus, we do not consider spaces of mixed dimension, such as the disjoint union of a plane and a line, to be manifolds at all. In Chapter 17, we will use the theory of de Rham cohomology to prove the following theorem, which shows that the dimension of a (nonempty) topological manifold is in fact a topological invariant.

Theorem 1.2 (Topological Invariance of Dimension). *A nonempty n-dimensional topological manifold cannot be homeomorphic to an m-dimensional manifold unless $m = n$.*

For the proof, see Theorem 17.26. In Chapter 2, we will also prove a related but weaker theorem (diffeomorphism invariance of dimension, Theorem 2.17). See also [LeeTM, Chap. 13] for a different proof of Theorem 1.2 using singular homology theory.

The empty set satisfies the definition of a topological n-manifold for every n. For the most part, we will ignore this special case (sometimes without remembering to say so). But because it is useful in certain contexts to allow the empty manifold, we choose not to exclude it from the definition.

The basic example of a topological n-manifold is $\mathbb{R}^n$ itself. It is Hausdorff because it is a metric space, and it is second-countable because the set of all open balls with rational centers and rational radii is a countable basis for its topology.

Requiring that manifolds share these properties helps to ensure that manifolds behave in the ways we expect from our experience with Euclidean spaces. For example, it is easy to verify that in a Hausdorff space, finite subsets are closed and limits of convergent sequences are unique (see Exercise A.11 in Appendix A). The motivation for second-countability is a bit less evident, but it will have important

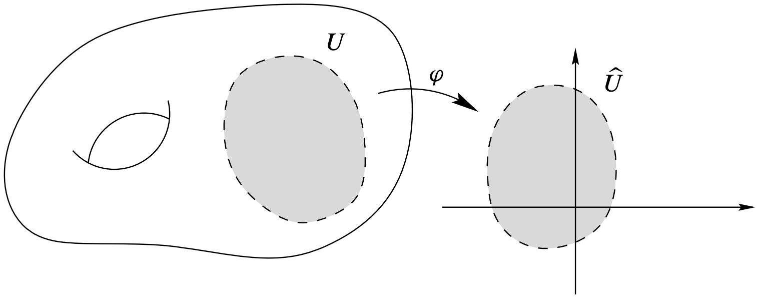

Fig. 1.2 A coordinate chart

consequences throughout the book, mostly based on the existence of partitions of unity (see Chapter 2).

In practice, both the Hausdorff and second-countability properties are usually easy to check, especially for spaces that are built out of other manifolds, because both properties are inherited by subspaces and finite products (Propositions A.17 and A.23). In particular, it follows that every open subset of a topological n-manifold is itself a topological n-manifold (with the subspace topology, of course).

We should note that some authors choose to omit the Hausdorff property or second-countability or both from the definition of manifolds. However, most of the interesting results about manifolds do in fact require these properties, and it is exceedingly rare to encounter a space "in nature" that would be a manifold except for the failure of one or the other of these hypotheses. For a couple of simple examples, see Problems 1-1 and 1-2; for a more involved example (a connected, locally Euclidean, Hausdorff space that is not second-countable), see [LeeTM, Problem 4-6].

Coordinate Charts

Let M be a topological n-manifold. A ***coordinate chart*** (or just a ***chart***) on M is a pair (U,φ), where U is an open subset of M and $\varphi\colon U \to \widehat{U}$ is a homeomorphism from U to an open subset $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$ (Fig. 1.2). By definition of a topological manifold, each point $p \in M$ is contained in the domain of some chart (U,φ). If $\varphi(p) = 0$, we say that the chart is ***centered at p***. If (U,φ) is any chart whose domain contains p, it is easy to obtain a new chart centered at p by subtracting the constant vector $\varphi(p)$.

Given a chart (U,φ), we call the set U a ***coordinate domain***, or a ***coordinate neighborhood*** of each of its points. If, in addition, $\varphi(U)$ is an open ball in $\mathbb{R}^n$, then U is called a ***coordinate ball***; if $\varphi(U)$ is an open cube, U is a ***coordinate cube***. The map φ is called a **(*local*) *coordinate map***, and the component functions $\left(x^1,\dots,x^n\right)$ of φ, defined by $\varphi(p) = \left(x^1(p),\dots,x^n(p)\right)$, are called ***local coordinates*** on U. We sometimes write things such as "(U,φ) is a chart containing p" as shorthand for "(U,φ) is a chart whose domain U contains p." If we wish to emphasize the coordinate functions $\left(x^1,\ldots,x^n\right)$ instead of the coordinate map φ, we sometimes denote the chart by $\left(U,\left(x^1,\ldots,x^n\right)\right)$ or $\left(U,\left(x^i\right)\right)$.

Examples of Topological Manifolds

Here are some simple examples.

Example 1.3 (Graphs of Continuous Functions). Let $U \subseteq \mathbb{R}^n$ be an open subset, and let $f\colon U \to \mathbb{R}^k$ be a continuous function. The ***graph of*** $\boldsymbol{f}$ is the subset of $\mathbb{R}^n \times \mathbb{R}^k$ defined by

$$\Gamma(f) = \left\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^k : x \in U \text{ and } y = f(x)\right\},$$

with the subspace topology. Let $\pi_1\colon \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ denote the projection onto the first factor, and let $\varphi\colon \Gamma(f) \to U$ be the restriction of π_1 to $\Gamma(f)$:

$$\varphi(x,y) = x, \quad (x,y) \in \Gamma(f).$$

Because φ is the restriction of a continuous map, it is continuous; and it is a homeomorphism because it has a continuous inverse given by $\varphi^{-1}(x) = (x, f(x))$. Thus $\Gamma(f)$ is a topological manifold of dimension n. In fact, $\Gamma(f)$ is homeomorphic to U itself, and $(\Gamma(f),\varphi)$ is a global coordinate chart, called ***graph coordinates***. The same observation applies to any subset of $\mathbb{R}^{n+k}$ defined by setting any k of the coordinates (not necessarily the last k) equal to some continuous function of the other n, which are restricted to lie in an open subset of $\mathbb{R}^n$. //

Example 1.4 (Spheres). For each integer $n \geq 0$, the unit n-sphere $\mathbb{S}^n$ is Hausdorff and second-countable because it is a topological subspace of $\mathbb{R}^{n+1}$. To show that it is locally Euclidean, for each index $i = 1,\ldots,n+1$ let U_i^+ denote the subset of $\mathbb{R}^{n+1}$ where the ith coordinate is positive:

$$U_i^+ = \left\{\left(x^1,\ldots,x^{n+1}\right) \in \mathbb{R}^{n+1} : x^i > 0\right\}.$$

(See Fig. 1.3.) Similarly, U_i^- is the set where $x^i < 0$.

Let $f\colon \mathbb{B}^n \to \mathbb{R}$ be the continuous function

$$f(u) = \sqrt{1-|u|^2}.$$

Then for each $i = 1,\ldots,n+1$, it is easy to check that $U_i^+ \cap \mathbb{S}^n$ is the graph of the function

$$x^i = f\left(x^1,\ldots,\widehat{x^i},\ldots,x^{n+1}\right),$$

where the hat indicates that x^i is omitted. Similarly, $U_i^- \cap \mathbb{S}^n$ is the graph of

$$x^i = -f\left(x^1,\ldots,\widehat{x^i},\ldots,x^{n+1}\right).$$

Thus, each subset $U_i^\pm \cap \mathbb{S}^n$ is locally Euclidean of dimension n, and the maps $\varphi_i^\pm\colon U_i^\pm \cap \mathbb{S}^n \to \mathbb{B}^n$ given by

$$\varphi_i^\pm\left(x^1,\ldots,x^{n+1}\right) = \left(x^1,\ldots,\widehat{x^i},\ldots,x^{n+1}\right)$$

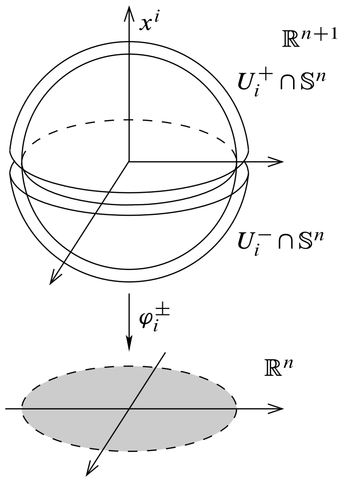

Fig. 1.3 Charts for $\mathbb{S}^n$

are graph coordinates for $\mathbb{S}^n$. Since each point of $\mathbb{S}^n$ is in the domain of at least one of these $2n+2$ charts, $\mathbb{S}^n$ is a topological n-manifold. //

Example 1.5 (Projective Spaces). The ***n-dimensional real projective space***, denoted by $\mathbb{RP}^n$ (or sometimes just $\mathbb{P}^n$), is defined as the set of 1-dimensional linear subspaces of $\mathbb{R}^{n+1}$, with the quotient topology determined by the natural map $\pi\colon \mathbb{R}^{n+1} \smallsetminus \{0\} \to \mathbb{RP}^n$ sending each point $x \in \mathbb{R}^{n+1} \smallsetminus \{0\}$ to the subspace spanned by x. The 2-dimensional projective space $\mathbb{RP}^2$ is called the ***projective plane***. For any point $x \in \mathbb{R}^{n+1} \smallsetminus \{0\}$, let $[x] = \pi(x) \in \mathbb{RP}^n$ denote the line spanned by x.

For each $i = 1, \dots, n+1$, let $\widetilde{U}_i \subseteq \mathbb{R}^{n+1} \smallsetminus \{0\}$ be the set where $x^i \neq 0$, and let $U_i = \pi\big(\widetilde{U}_i\big) \subseteq \mathbb{RP}^n$. Since $\widetilde{U}_i$ is a saturated open subset, U_i is open and $\pi|_{\widetilde{U}_i}\colon \widetilde{U}_i \to U_i$ is a quotient map (see Theorem A.27). Define a map $\varphi_i\colon U_i \to \mathbb{R}^n$ by

$$\varphi_i\left[x^1, \dots, x^{n+1}\right] = \left(\frac{x^1}{x^i}, \dots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \dots, \frac{x^{n+1}}{x^i}\right).$$

This map is well defined because its value is unchanged by multiplying x by a nonzero constant. Because $\varphi_i \circ \pi$ is continuous, φ_i is continuous by the characteristic property of quotient maps (Theorem A.27). In fact, φ_i is a homeomorphism, because it has a continuous inverse given by

$$\varphi_i^{-1}\left(u^1, \dots, u^n\right) = \left[u^1, \dots, u^{i-1}, 1, u^i, \dots, u^n\right],$$

as you can check. Geometrically, $\varphi([x]) = u$ means $(u, 1)$ is the point in $\mathbb{R}^{n+1}$ where the line $[x]$ intersects the affine hyperplane where $x^i = 1$ (Fig. 1.4). Because the sets $U_1, \dots, U_{n+1}$ cover $\mathbb{RP}^n$, this shows that $\mathbb{RP}^n$ is locally Euclidean of dimension n. The Hausdorff and second-countability properties are left as exercises. //

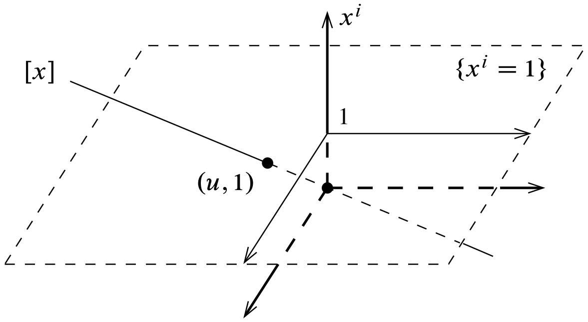

Fig. 1.4 A chart for $\mathbb{RP}^n$

► **Exercise 1.6.** Show that $\mathbb{RP}^n$ is Hausdorff and second-countable, and is therefore a topological n-manifold.

► **Exercise 1.7.** Show that $\mathbb{RP}^n$ is compact. [Hint: show that the restriction of π to $\mathbb{S}^n$ is surjective.]

Example 1.8 (Product Manifolds). Suppose $M_1, \dots, M_k$ are topological manifolds of dimensions $n_1, \dots, n_k$, respectively. The product space $M_1 \times \cdots \times M_k$ is shown to be a topological manifold of dimension $n_1 + \cdots + n_k$ as follows. It is Hausdorff and second-countable by Propositions A.17 and A.23, so only the locally Euclidean property needs to be checked. Given any point $(p_1, \dots, p_k) \in M_1 \times \cdots \times M_k$, we can choose a coordinate chart (U_i, φ_i) for each M_i with $p_i \in U_i$. The product map

$$\varphi_1 \times \cdots \times \varphi_k \colon U_1 \times \cdots \times U_k \to \mathbb{R}^{n_1 + \cdots + n_k}$$

is a homeomorphism onto its image, which is a product open subset of $\mathbb{R}^{n_1 + \cdots + n_k}$. Thus, $M_1 \times \cdots \times M_k$ is a topological manifold of dimension $n_1 + \cdots + n_k$, with charts of the form $(U_1 \times \cdots \times U_k, \varphi_1 \times \cdots \times \varphi_k)$. //

Example 1.9 (Tori). For a positive integer n, the ***n-torus*** (plural: ***tori***) is the product space $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$. By the discussion above, it is a topological n-manifold. (The 2-torus is usually called simply ***the torus***.) //

Topological Properties of Manifolds

As topological spaces go, manifolds are quite special, because they share so many important properties with Euclidean spaces. Here we discuss a few such properties that will be of use to us throughout the book.

Most of the properties we discuss in this section depend on the fact that every manifold possesses a particularly well-behaved basis for its topology.

Lemma 1.10. *Every topological manifold has a countable basis of precompact coordinate balls.*

Proof. Let M be a topological n-manifold. First we consider the special case in which M can be covered by a single chart. Suppose $\varphi\colon M \to \widehat{U} \subseteq \mathbb{R}^n$ is a global coordinate map, and let $\mathcal{B}$ be the collection of all open balls $B_r(x) \subseteq \mathbb{R}^n$ such that r is rational, x has rational coordinates, and $B_{r'}(x) \subseteq \widehat{U}$ for some $r' > r$. Each such ball is precompact in $\widehat{U}$, and it is easy to check that $\mathcal{B}$ is a countable basis for the topology of $\widehat{U}$. Because φ is a homeomorphism, it follows that the collection of sets of the form $\varphi^{-1}(B)$ for $B \in \mathcal{B}$ is a countable basis for the topology of M, consisting of precompact coordinate balls, with the restrictions of φ as coordinate maps.

Now let M be an arbitrary n-manifold. By definition, each point of M is in the domain of a chart. Because every open cover of a second-countable space has a countable subcover (Proposition A.16), M is covered by countably many charts $\{(U_i, \varphi_i)\}$. By the argument in the preceding paragraph, each coordinate domain U_i has a countable basis of coordinate balls that are precompact in U_i, and the union of all these countable bases is a countable basis for the topology of M. If $V \subseteq U_i$ is one of these balls, then the closure of V in U_i is compact, and because M is Hausdorff, it is closed in M. It follows that the closure of V in M is the same as its closure in U_i, so V is precompact in M as well. □

Connectivity

The existence of a basis of coordinate balls has important consequences for the connectivity properties of manifolds. Recall that a topological space X is

- ***connected*** if there do not exist two disjoint, nonempty, open subsets of X whose union is X;
- ***path-connected*** if every pair of points in X can be joined by a path in X; and
- ***locally path-connected*** if X has a basis of path-connected open subsets.

(See Appendix A.) The following proposition shows that connectivity and path connectivity coincide for manifolds.

Proposition 1.11. *Let M be a topological manifold.*

(a) *M is locally path-connected.*
(b) *M is connected if and only if it is path-connected.*
(c) *The components of M are the same as its path components.*
(d) *M has countably many components, each of which is an open subset of M and a connected topological manifold.*

Proof. Since each coordinate ball is path-connected, (a) follows from the fact that M has a basis of coordinate balls. Parts (b) and (c) are immediate consequences of (a) and Proposition A.43. To prove (d), note that each component is open in M by Proposition A.43, so the collection of components is an open cover of M. Because M is second-countable, this cover must have a countable subcover. But since the components are all disjoint, the cover must have been countable to begin with, which is to say that M has only countably many components. Because the components are open, they are connected topological manifolds in the subspace topology. □

Local Compactness and Paracompactness

The next topological property of manifolds that we need is local compactness (see Appendix A for the definition).

Proposition 1.12 (Manifolds Are Locally Compact). *Every topological manifold is locally compact.*

Proof. Lemma 1.10 showed that every manifold has a basis of precompact open subsets. □

Another key topological property possessed by manifolds is called *paracompactness*. It is a consequence of local compactness and second-countability, and in fact is one of the main reasons why second-countability is included in the definition of manifolds.

Let M be a topological space. A collection $\mathcal{X}$ of subsets of M is said to be ***locally finite*** if each point of M has a neighborhood that intersects at most finitely many of the sets in $\mathcal{X}$. Given a cover $\mathcal{U}$ of M, another cover $\mathcal{V}$ is called a ***refinement of*** $\boldsymbol{\mathcal{U}}$ if for each $V \in \mathcal{V}$ there exists some $U \in \mathcal{U}$ such that $V \subseteq U$. We say that M is ***paracompact*** if every open cover of M admits an open, locally finite refinement.

Lemma 1.13. *Suppose $\mathcal{X}$ is a locally finite collection of subsets of a topological space M.*

(a) *The collection $\{\bar{X} : X \in \mathcal{X}\}$ is also locally finite.*
(b) $\overline{\bigcup_{X \in \mathcal{X}} X} = \bigcup_{X \in \mathcal{X}} \bar{X}$.

► **Exercise 1.14.** Prove the preceding lemma.

Theorem 1.15 (Manifolds Are Paracompact). *Every topological manifold is paracompact. In fact, given a topological manifold M, an open cover $\mathcal{X}$ of M, and any basis $\mathcal{B}$ for the topology of M, there exists a countable, locally finite open refinement of $\mathcal{X}$ consisting of elements of $\mathcal{B}$.*

Proof. Given M, $\mathcal{X}$, and $\mathcal{B}$ as in the hypothesis of the theorem, let $(K_j)_{j=1}^{\infty}$ be an exhaustion of M by compact sets (Proposition A.60). For each j, let $V_j = K_{j+1} \smallsetminus \operatorname{Int} K_j$ and $W_j = \operatorname{Int} K_{j+2} \smallsetminus K_{j-1}$ (where we interpret K_j as $\varnothing$ if $j < 1$). Then V_j is a compact set contained in the open subset W_j. For each $x \in V_j$, there is some $X_x \in \mathcal{X}$ containing x, and because $\mathcal{B}$ is a basis, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq X_x \cap W_j$. The collection of all such sets B_x as x ranges over V_j is an open cover of V_j, and thus has a finite subcover. The union of all such finite subcovers as j ranges over the positive integers is a countable open cover of M that refines $\mathcal{X}$. Because the finite subcover of V_j consists of sets contained in W_j, and $W_j \cap W_{j'} = \varnothing$ except when $j - 2 \le j' \le j + 2$, the resulting cover is locally finite. □

Problem 1-5 shows that, at least for connected spaces, paracompactness can be used as a substitute for second-countability in the definition of manifolds.

Fundamental Groups of Manifolds

The following result about fundamental groups of manifolds will be important in our study of covering manifolds in Chapter 4. For a brief review of the fundamental group, see Appendix A.

Proposition 1.16. *The fundamental group of a topological manifold is countable.*

Proof. Let M be a topological manifold. By Lemma 1.10, there is a countable collection $\mathcal{B}$ of coordinate balls covering M. For any pair of coordinate balls $B, B' \in \mathcal{B}$, the intersection $B \cap B'$ has at most countably many components, each of which is path-connected. Let $\mathcal{X}$ be a countable set containing a point from each component of $B \cap B'$ for each $B, B' \in \mathcal{B}$ (including $B = B'$). For each $B \in \mathcal{B}$ and each $x, x' \in \mathcal{X}$ such that $x, x' \in B$, let $h^B_{x,x'}$ be some path from x to x' in B.

Since the fundamental groups based at any two points in the same component of M are isomorphic, and $\mathcal{X}$ contains at least one point in each component of M, we may as well choose a point $p \in \mathcal{X}$ as base point. Define a ***special loop*** to be a loop based at p that is equal to a finite product of paths of the form $h^B_{x,x'}$. Clearly, the set of special loops is countable, and each special loop determines an element of $\pi_1(M, p)$. To show that $\pi_1(M, p)$ is countable, therefore, it suffices to show that each element of $\pi_1(M, p)$ is represented by a special loop.

Suppose $f : [0,1] \to M$ is a loop based at p. The collection of components of sets of the form $f^{-1}(B)$ as B ranges over $\mathcal{B}$ is an open cover of $[0,1]$, so by compactness it has a finite subcover. Thus, there are finitely many numbers $0 = a_0 < a_1 < \cdots < a_k = 1$ such that $[a_{i-1}, a_i] \subseteq f^{-1}(B)$ for some $B \subseteq \mathcal{B}$. For each i, let f_i be the restriction of f to the interval $[a_{i-1}, a_i]$, reparametrized so that its domain is $[0,1]$, and let $B_i \in \mathcal{B}$ be a coordinate ball containing the image of f_i. For each i, we have $f(a_i) \in B_i \cap B_{i+1}$, and there is some $x_i \in \mathcal{X}$ that lies in the same component of $B_i \cap B_{i+1}$ as $f(a_i)$. Let g_i be a path in $B_i \cap B_{i+1}$ from x_i to $f(a_i)$ (Fig. 1.5), with the understanding that $x_0 = x_k = p$, and g_0 and g_k are both equal to the constant path c_p based at p. Then, because $\overline{g}_i \cdot g_i$ is path-homotopic to a constant path (where $\overline{g}_i(t) = g_i(1-t)$ is the reverse path of g_i),

$$\begin{aligned} f &\sim f_1 \cdot \cdots \cdot f_k \\ &\sim g_0 \cdot f_1 \cdot \overline{g}_1 \cdot g_1 \cdot f_2 \cdot \overline{g}_2 \cdot \cdots \cdot \overline{g}_{k-1} \cdot g_{k-1} \cdot f_k \cdot \overline{g}_k \\ &\sim \widetilde{f}_1 \cdot \widetilde{f}_2 \cdot \cdots \cdot \widetilde{f}_k, \end{aligned}$$

where $\widetilde{f}_i = g_{i-1} \cdot f_i \cdot \overline{g}_i$. For each i, $\widetilde{f}_i$ is a path in B_i from x_{i-1} to x_i. Since B_i is simply connected, $\widetilde{f}_i$ is path-homotopic to $h^{B_i}_{x_{i-1},x_i}$. It follows that f is path-homotopic to a special loop, as claimed. □

Smooth Structures

The definition of manifolds that we gave in the preceding section is sufficient for studying topological properties of manifolds, such as compactness, connectedness,

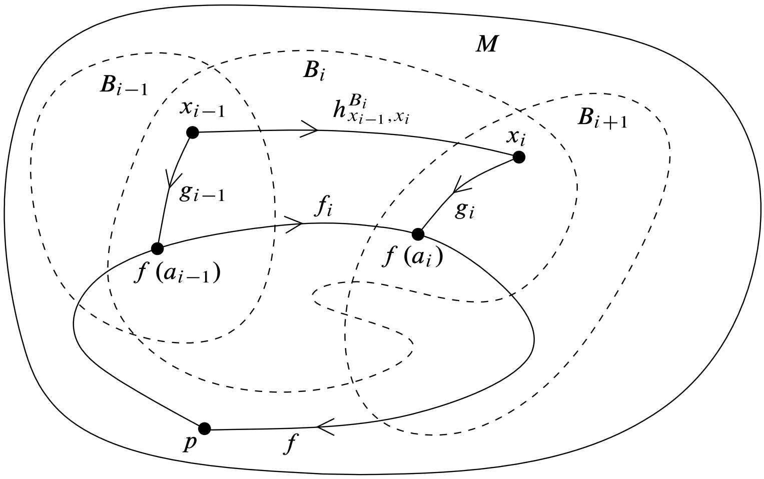

Fig. 1.5 The fundamental group of a manifold is countable

simple connectivity, and the problem of classifying manifolds up to homeomorphism. However, in the entire theory of topological manifolds there is no mention of calculus. There is a good reason for this: however we might try to make sense of derivatives of functions on a manifold, such derivatives cannot be invariant under homeomorphisms. For example, the map $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $\varphi(u,v) = \left(u^{1/3}, v^{1/3}\right)$ is a homeomorphism, and it is easy to construct differentiable functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $f \circ \varphi$ is not differentiable at the origin. (The function $f(x,y) = x$ is one such.)

To make sense of derivatives of real-valued functions, curves, or maps between manifolds, we need to introduce a new kind of manifold called a *smooth manifold*. It will be a topological manifold with some extra structure in addition to its topology, which will allow us to decide which functions to or from the manifold are smooth.

The definition will be based on the calculus of maps between Euclidean spaces, so let us begin by reviewing some basic terminology about such maps. If U and V are open subsets of Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, a function $F\colon U \to V$ is said to be ***smooth*** (or $\boldsymbol{C^\infty}$, or ***infinitely differentiable***) if each of its component functions has continuous partial derivatives of all orders. If in addition F is bijective and has a smooth inverse map, it is called a ***diffeomorphism***. A diffeomorphism is, in particular, a homeomorphism.

A review of some important properties of smooth maps is given in Appendix C. You should be aware that some authors define the word *smooth* differently—for example, to mean continuously differentiable or merely differentiable. On the other hand, some use the word *differentiable* to mean what we call *smooth*. Throughout this book, *smooth* is synonymous with C^∞.

To see what additional structure on a topological manifold might be appropriate for discerning which maps are smooth, consider an arbitrary topological n-manifold M. Each point in M is in the domain of a coordinate map $\varphi\colon U \to \widehat{U} \subseteq \mathbb{R}^n$.

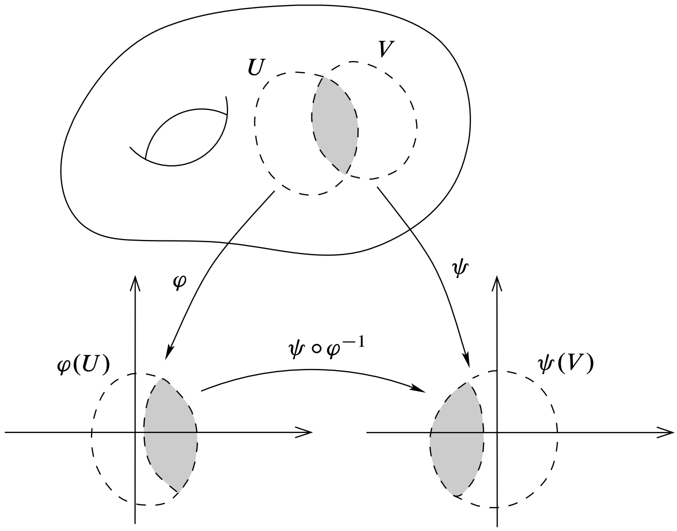

Fig. 1.6 A transition map

A plausible definition of a smooth function on M would be to say that $f: M \to \mathbb{R}$ is smooth if and only if the composite function $f \circ \varphi^{-1}: \widehat{U} \to \mathbb{R}$ is smooth in the sense of ordinary calculus. But this will make sense only if this property is independent of the choice of coordinate chart. To guarantee this independence, we will restrict our attention to "smooth charts." Since smoothness is not a homeomorphism-invariant property, the way to do this is to consider the collection of all smooth charts as a new kind of structure on M.

With this motivation in mind, we now describe the details of the construction.

Let M be a topological n-manifold. If (U, φ), (V, ψ) are two charts such that $U \cap V \neq \varnothing$, the composite map $\psi \circ \varphi^{-1}: \varphi(U \cap V) \to \psi(U \cap V)$ is called the ***transition map from φ to ψ*** (Fig. 1.6). It is a composition of homeomorphisms, and is therefore itself a homeomorphism. Two charts (U, φ) and (V, ψ) are said to be ***smoothly compatible*** if either $U \cap V = \varnothing$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism. Since $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open subsets of $\mathbb{R}^n$, smoothness of this map is to be interpreted in the ordinary sense of having continuous partial derivatives of all orders.

We define an ***atlas for M*** to be a collection of charts whose domains cover M. An atlas $\mathcal{A}$ is called a ***smooth atlas*** if any two charts in $\mathcal{A}$ are smoothly compatible with each other.

To show that an atlas is smooth, we need only verify that each transition map $\psi \circ \varphi^{-1}$ is smooth whenever (U, φ) and (V, ψ) are charts in $\mathcal{A}$; once we have proved this, it follows that $\psi \circ \varphi^{-1}$ is a diffeomorphism because its inverse $\left(\psi \circ \varphi^{-1}\right)^{-1} = \varphi \circ \psi^{-1}$ is one of the transition maps we have already shown to be smooth. Alternatively, given two particular charts (U, φ) and (V, ψ), it is often easiest to show that they are smoothly compatible by verifying that $\psi \circ \varphi^{-1}$ is smooth and injective with nonsingular Jacobian at each point, and appealing to Corollary C.36.

Our plan is to define a "smooth structure" on M by giving a smooth atlas, and to define a function $f: M \to \mathbb{R}$ to be smooth if and only if $f \circ \varphi^{-1}$ is smooth in the sense of ordinary calculus for each coordinate chart (U, φ) in the atlas. There is one minor technical problem with this approach: in general, there will be many possible atlases that give the "same" smooth structure, in that they all determine the same collection of smooth functions on M. For example, consider the following pair of atlases on $\mathbb{R}^n$:

$$\mathcal{A}_1 = \left\{ \left(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n} \right) \right\},$$
$$\mathcal{A}_2 = \left\{ \left(B_1(x), \mathrm{Id}_{B_1(x)} \right) : x \in \mathbb{R}^n \right\}.$$

Although these are different smooth atlases, clearly a function $f: \mathbb{R}^n \to \mathbb{R}$ is smooth with respect to either atlas if and only if it is smooth in the sense of ordinary calculus.

We could choose to define a smooth structure as an equivalence class of smooth atlases under an appropriate equivalence relation. However, it is more straightforward to make the following definition: a smooth atlas $\mathcal{A}$ on M is ***maximal*** if it is not properly contained in any larger smooth atlas. This just means that any chart that is smoothly compatible with every chart in $\mathcal{A}$ is already in $\mathcal{A}$. (Such a smooth atlas is also said to be ***complete***.)

Now we can define the main concept of this chapter. If M is a topological manifold, a ***smooth structure on M*** is a maximal smooth atlas. A ***smooth manifold*** is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M. When the smooth structure is understood, we usually omit mention of it and just say "M is a smooth manifold." Smooth structures are also called ***differentiable structures*** or ***C^∞ structures*** by some authors. We also use the term ***smooth manifold structure*** to mean a manifold topology together with a smooth structure.

We emphasize that a smooth structure is an additional piece of data that must be added to a topological manifold before we are entitled to talk about a "smooth manifold." In fact, a given topological manifold may have many different smooth structures (see Example 1.23 and Problem 1-6). On the other hand, it is not always possible to find a smooth structure on a given topological manifold: there exist topological manifolds that admit no smooth structures at all. (The first example was a compact 10-dimensional manifold found in 1960 by Michel Kervaire [Ker60].)

It is generally not very convenient to define a smooth structure by explicitly describing a maximal smooth atlas, because such an atlas contains very many charts. Fortunately, we need only specify *some* smooth atlas, as the next proposition shows.

Proposition 1.17. *Let M be a topological manifold.*

(a) *Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas, called the* ***smooth structure determined by $\mathcal{A}$***.

(b) *Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.*

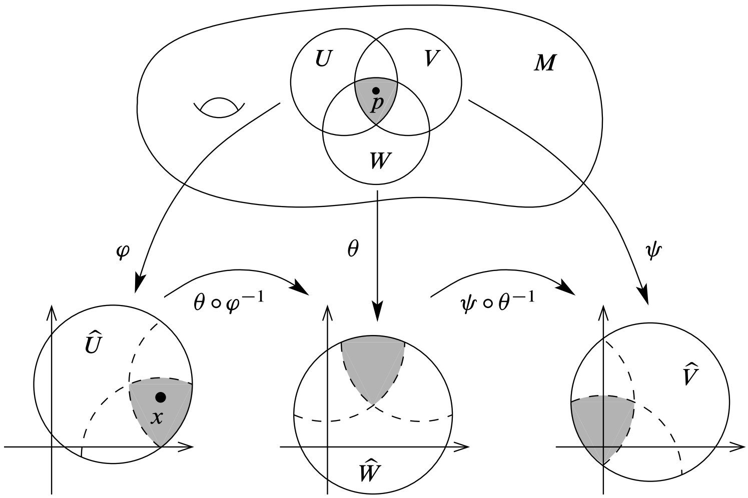

Fig. 1.7 Proof of Proposition 1.17(a)

Proof. Let $\mathcal{A}$ be a smooth atlas for M, and let $\overline{\mathcal{A}}$ denote the set of all charts that are smoothly compatible with every chart in $\mathcal{A}$. To show that $\overline{\mathcal{A}}$ is a smooth atlas, we need to show that any two charts of $\overline{\mathcal{A}}$ are smoothly compatible with each other, which is to say that for any (U,φ), $(V,\psi) \in \overline{\mathcal{A}}$, the map $\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$ is smooth.

Let $x = \varphi(p) \in \varphi(U \cap V)$ be arbitrary. Because the domains of the charts in $\mathcal{A}$ cover M, there is some chart $(W,\theta) \in \mathcal{A}$ such that $p \in W$ (Fig. 1.7). Since every chart in $\overline{\mathcal{A}}$ is smoothly compatible with (W,θ), both of the maps $\theta \circ \varphi^{-1}$ and $\psi \circ \theta^{-1}$ are smooth where they are defined. Since $p \in U \cap V \cap W$, it follows that $\psi \circ \varphi^{-1} = \left(\psi \circ \theta^{-1}\right) \circ \left(\theta \circ \varphi^{-1}\right)$ is smooth on a neighborhood of x. Thus, $\psi \circ \varphi^{-1}$ is smooth in a neighborhood of each point in $\varphi(U \cap V)$. Therefore, $\overline{\mathcal{A}}$ is a smooth atlas. To check that it is maximal, just note that any chart that is smoothly compatible with every chart in $\overline{\mathcal{A}}$ must in particular be smoothly compatible with every chart in $\mathcal{A}$, so it is already in $\overline{\mathcal{A}}$. This proves the existence of a maximal smooth atlas containing $\mathcal{A}$. If $\mathcal{B}$ is any other maximal smooth atlas containing $\mathcal{A}$, each of its charts is smoothly compatible with each chart in $\mathcal{A}$, so $\mathcal{B} \subseteq \overline{\mathcal{A}}$. By maximality of $\mathcal{B}$, $\mathcal{B} = \overline{\mathcal{A}}$.

The proof of (b) is left as an exercise. □

► **Exercise 1.18.** Prove Proposition 1.17(b).

For example, if a topological manifold M can be covered by a single chart, the smooth compatibility condition is trivially satisfied, so any such chart automatically determines a smooth structure on M.

It is worth mentioning that the notion of smooth structure can be generalized in several different ways by changing the compatibility requirement for charts. For example, if we replace the requirement that charts be smoothly compatible by the weaker requirement that each transition map $\psi \circ \varphi^{-1}$ (and its inverse) be of class C^k, we obtain the definition of a ***C^k structure***. Similarly, if we require that each transition map be real-analytic (i.e., expressible as a convergent power series in a neighborhood of each point), we obtain the definition of a ***real-analytic structure***, also called a ***C^ω structure***. If M has even dimension $n = 2m$, we can identify $\mathbb{R}^{2m}$ with $\mathbb{C}^m$ and require that the transition maps be complex-analytic; this determines a ***complex-analytic structure***. A manifold endowed with one of these structures is called a ***C^k manifold***, ***real-analytic manifold***, or ***complex manifold***, respectively. (Note that a C^0 manifold is just a topological manifold.) We do not treat any of these other kinds of manifolds in this book, but they play important roles in analysis, so it is useful to know the definitions.

Local Coordinate Representations

If M is a smooth manifold, any chart (U, φ) contained in the given maximal smooth atlas is called a ***smooth chart***, and the corresponding coordinate map φ is called a ***smooth coordinate map***. It is useful also to introduce the terms ***smooth coordinate domain*** or ***smooth coordinate neighborhood*** for the domain of a smooth coordinate chart. A ***smooth coordinate ball*** means a smooth coordinate domain whose image under a smooth coordinate map is a ball in Euclidean space. A ***smooth coordinate cube*** is defined similarly.

It is often useful to restrict attention to coordinate balls whose closures sit nicely inside larger coordinate balls. We say a set $B \subseteq M$ is a ***regular coordinate ball*** if there is a smooth coordinate ball $B' \supseteq \overline{B}$ and a smooth coordinate map $\varphi\colon B' \to \mathbb{R}^n$ such that for some positive real numbers $r < r'$,

$$\varphi(B) = B_r(0), \qquad \varphi\big(\overline{B}\big) = \overline{B}_r(0), \quad \text{and} \quad \varphi\left(B'\right) = B_{r'}(0).$$

Because $\overline{B}$ is homeomorphic to $\overline{B}_r(0)$, it is compact, and thus every regular coordinate ball is precompact in M. The next proposition gives a slight improvement on Lemma 1.10 for smooth manifolds. Its proof is a straightforward adaptation of the proof of that lemma.

Proposition 1.19. *Every smooth manifold has a countable basis of regular coordinate balls.*

► **Exercise 1.20.** Prove Proposition 1.19.

Here is how one usually thinks about coordinate charts on a smooth manifold. Once we choose a smooth chart (U, φ) on M, the coordinate map $\varphi\colon U \to \widehat{U} \subseteq \mathbb{R}^n$ can be thought of as giving a temporary *identification* between U and $\widehat{U}$. Using this identification, while we work in this chart, we can think of U simultaneously as an open subset of M and as an open subset of $\mathbb{R}^n$. You can visualize this identification by thinking of a "grid" drawn on U representing the preimages of the coordinate lines under φ (Fig. 1.8). Under this identification, we can represent a point $p \in U$ by its coordinates $\left(x^1, \dots, x^n\right) = \varphi(p)$, and think of this n-tuple as *being* the

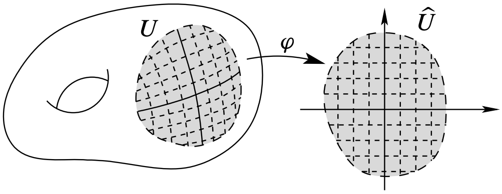

Fig. 1.8 A coordinate grid

point p. We typically express this by saying "$\left(x^1, \ldots, x^n\right)$ is the (local) coordinate representation for p" or "$p = \left(x^1, \ldots, x^n\right)$ in local coordinates."

Another way to look at it is that by means of our identification $U \leftrightarrow \widehat{U}$, we can think of φ as the identity map and suppress it from the notation. This takes a bit of getting used to, but the payoff is a huge simplification of the notation in many situations. You just need to remember that the identification is in general only local, and depends heavily on the choice of coordinate chart.

You are probably already used to such identifications from your study of multivariable calculus. The most common example is ***polar coordinates*** (r, θ) in the plane, defined implicitly by the relation $(x, y) = (r\cos\theta, r\sin\theta)$ (see Example C.37). On an appropriate open subset such as $U = \{(x, y) : x > 0\} \subseteq \mathbb{R}^2$, (r, θ) can be expressed as smooth functions of (x, y), and the map that sends (x, y) to the corresponding (r, θ) is a smooth coordinate map with respect to the standard smooth structure on $\mathbb{R}^2$. Using this map, we can write a given point $p \in U$ either as $p = (x, y)$ in standard coordinates or as $p = (r, \theta)$ in polar coordinates, where the two coordinate representations are related by $(r, \theta) = \left(\sqrt{x^2 + y^2}, \tan^{-1} y/x\right)$ and $(x, y) = (r\cos\theta, r\sin\theta)$. Other polar coordinate charts can be obtained by restricting (r, θ) to other open subsets of $\mathbb{R}^2 \smallsetminus \{0\}$.

The fact that manifolds do not come with any predetermined choice of coordinates is both a blessing and a curse. The flexibility to choose coordinates more or less arbitrarily can be a big advantage in approaching problems in manifold theory, because the coordinates can often be chosen to simplify some aspect of the problem at hand. But we pay for this flexibility by being obliged to ensure that any objects we wish to define globally on a manifold are not dependent on a particular choice of coordinates. There are generally two ways of doing this: either by writing down a coordinate-dependent definition and then proving that the definition gives the same results in any coordinate chart, or by writing down a definition that is manifestly coordinate-independent (often called an *invariant definition*). We will use the coordinate-dependent approach in a few circumstances where it is notably simpler, but for the most part we will give coordinate-free definitions whenever possible. The need for such definitions accounts for much of the abstraction of modern manifold theory. One of the most important skills you will need to acquire in order to use manifold theory effectively is an ability to switch back and forth easily between invariant descriptions and their coordinate counterparts.

Examples of Smooth Manifolds

Before proceeding further with the general theory, let us survey some examples of smooth manifolds.

Example 1.21 (0-Dimensional Manifolds). A topological manifold M of dimension 0 is just a countable discrete space. For each point $p \in M$, the only neighborhood of p that is homeomorphic to an open subset of $\mathbb{R}^0$ is $\{p\}$ itself, and there is exactly one coordinate map $\varphi\colon \{p\} \to \mathbb{R}^0$. Thus, the set of all charts on M trivially satisfies the smooth compatibility condition, and each 0-dimensional manifold has a unique smooth structure. //

Example 1.22 (Euclidean Spaces). For each nonnegative integer n, the Euclidean space $\mathbb{R}^n$ is a smooth n-manifold with the smooth structure determined by the atlas consisting of the single chart $(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})$. We call this the ***standard smooth structure on*** $\mathbb{R}^n$ and the resulting coordinate map ***standard coordinates***. Unless we explicitly specify otherwise, we always use this smooth structure on $\mathbb{R}^n$. With respect to this smooth structure, the smooth coordinate charts for $\mathbb{R}^n$ are exactly those charts (U, φ) such that φ is a diffeomorphism (in the sense of ordinary calculus) from U to another open subset $\widehat{U} \subseteq \mathbb{R}^n$. //

Example 1.23 (Another Smooth Structure on $\mathbb{R}$). Consider the homeomorphism $\psi\colon \mathbb{R} \to \mathbb{R}$ given by

$$\psi(x) = x^3. \tag{1.1}$$

The atlas consisting of the single chart $(\mathbb{R}, \psi)$ defines a smooth structure on $\mathbb{R}$. This chart is not smoothly compatible with the standard smooth structure, because the transition map $\mathrm{Id}_{\mathbb{R}} \circ \psi^{-1}(y) = y^{1/3}$ is not smooth at the origin. Therefore, the smooth structure defined on $\mathbb{R}$ by ψ is not the same as the standard one. Using similar ideas, it is not hard to construct many distinct smooth structures on any given positive-dimensional topological manifold, as long as it has one smooth structure to begin with (see Problem 1-6). //

Example 1.24 (Finite-Dimensional Vector Spaces). Let V be a finite-dimensional real vector space. Any norm on V determines a topology, which is independent of the choice of norm (Exercise B.49). With this topology, V is a topological n-manifold, and has a natural smooth structure defined as follows. Each (ordered) basis $(E_1, \dots, E_n)$ for V defines a basis isomorphism $E\colon \mathbb{R}^n \to V$ by

$$E(x) = \sum_{i=1}^{n} x^i E_i.$$

This map is a homeomorphism, so $\left(V, E^{-1}\right)$ is a chart. If $\left(\widetilde{E}_1, \dots, \widetilde{E}_n\right)$ is any other basis and $\widetilde{E}(x) = \sum_j x^j \widetilde{E}_j$ is the corresponding isomorphism, then there is some invertible matrix $\left(A_i^j\right)$ such that $E_i = \sum_j A_i^j \widetilde{E}_j$ for each i. The transition map between the two charts is then given by $\widetilde{E}^{-1} \circ E(x) = \widetilde{x}$, where $\widetilde{x} = \left(\widetilde{x}^1, \dots, \widetilde{x}^n\right)$