A Theoretical Geometric Model for Olfactory Learning and Sensory Processing

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Abstract

Using the tools of smooth manifold theory, we propose a generalized framework for olfactory reception, learning, and processing. Inspection of the tangent bundle to a manifold yields vector fields which allow for quantification of changes. We utilize group actions to discover fibre bundles over the manifold and discover various properties related to learning. Under this paradigm, we develop a method for categorization as well as analytical tools to model changes in the category. We end with a quick discussion of searching for data on the manifold in a way that beats "nearest neighbour."

Introduction

The current view of scent reception and processing relies mainly on biological explanations of neuronal processes and molecular analysis. Little has been done to explain olfaction through rigorous mathematical analysis different from statistical methods. In contrast, other modalities, such as vision, have been extensively analyzed and explored in the most abstract reference frame due to some convenient properties such as invariants. This exploration will walk through a theory of the mathematical basis for olfactory reception, learning, and processing. This will culminate in the construction of a generalized framework for olfactory representation and learning.

No discussion of such a topic can be done without the study of manifolds, particularly smooth and riemannian manifolds, in the category of differentiable topological spaces. So, we present the following definitions.[1]

Definition 2.1. A set X is called a topological space if it is equipped with a collection of open sets, τ, such that for any arbitrary indexing set I and a finite indexing set J, and $U_k \subseteq X, k \in \{i, j\}$

1. $\bigcup_{i \in I} U_i \in \tau$
2. $\bigcap_{j \in J} U_j \in \tau$

[1] Additional definitions and various notation can be found in Appendix B and C.

3. $\varnothing, X \in \tau$

τ is called the topology on X and defines the open sets of X. We denote this space, (X, τ). If there is no ambiguity about the topology we simply write, X.

Topological spaces are the basic constructions we are interested in. In order to talk about continuous or differentiable structures we first need a well defined topology on the space.[2] Manifolds are an example of topological spaces which carry additional properties which make them particularly "nice" to work with. They will form the basis of the rest of this analysis.

Definition 2.2. A Topological $n-$manifold is a topological space such that the following hold:

1. M is Hausdorff
2. M is second countable
3. M is locally Euclidean of dimension n

Definition 2.3. A topological space X, is Hausdorff if and only if for any two points, $x, y \in X$, there exist, $U, V \subsetneq X$ such that $x \in U, y \in V$ and $U \cap V = \varnothing$.

This property allows us to separate any two non-identical points into disjoint regions. This becomes key in any identification task as we need a method of discrimination completely independent of the actual inputs. Suppose we have two pairs of inputs, P_1 and P_2. We need a generalizable method to distinguish the elements of the pairs as well as the pairs themselves. Furthermore, we invoke the locally connected property of manifolds of dimension $\dim M \geq 1$. This will allow us to define certain metrics on the spaces, giving more concrete methods of separation and identification.

Notice that this definition alone does not allow for us to do calculus on any given manifold. To allow for such constructions, we need the idea of a smooth structure[3]. With this smooth strucure we can now construct a manifold in the category of differetiable spaces, called a differetiable manifold, denoted $C^\infty-$manifold.

The differentiable manifolds we will use in this paper are all Riemannian Manifolds. Before we give the definition of these, we need to define the tangent space to a manifold.

[2] See Appendix B

[3] See section on M-space

Definition 2.4. Let M be a smooth manifold and $C_p = \{c : (-\epsilon, \epsilon) \to M | c(0) = p\}$ be the set of all equivlance classes of paths passing through $p \in M$. The Tangent Space, denoted T_pM, of M at p is the collection

$$T_pM = \{c'(0) | c(t) \in C_p\}.$$

That is to say, it is the collection of all velocity vectors of differentiable curves passing through p.

A classical example of such a construction is the tangent space to the unit circle, $\mathbb{S}^1 = \{e^{i\theta} | \theta \in \mathbb{R}\}$ at the point $\theta = 0$. It is obvious that this should just be the vertical line, $x = 1$.

Definition 2.5. A Riemannian Manifold, R is a C^∞-topological manifold which has, at each point $p \in R$ a well defined inner product

$$\langle \cdot, \cdot \rangle_p : T_pR \times T_pR \to \mathbb{R}$$

on the tangent space, T_pR which varies smoothly from point to point.[4]

We use Riemannian Manifolds because they arrise naturally when discussing angles on manifolds. Locally, for any two vectors, the angle between them will be the same as it is in the tangent space. The inner product operation, $\langle \cdot, \cdot \rangle$ defines the angle between two vectors in the tangent space.

Furthermore, we will briefly discuss the idea of "sameness". To a mathematician there are many different types of equivalent objects. To an algebrist, sameness is the isomorphism. As our investigation is topological in nature, our definition of "same" is homeomorphism and diffeomorphism.

Definition 2.6. Let X, Y be two manifolds. X and Y are C^k-diffeomorphic (denoted $X \overset{d}{\cong} Y$) if and only if there exists a bijective C^k-function [5] $D : X \to Y$ which has a C^k- inverse.

Homeomorphism is a special (but important) case where $k = 0$, so the functions are simply continuous. For example, take the full square, $\left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ it is homeomorphic to a disk, $\overline{\mathbb{B}}^2 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$. These two are not diffeomorphic because the corners of the square are not differentiable.

[4] Notice that in the definition we assume that the tangent spaces are real vector spaces. If instead T_pR is a complex vector space, we can treat it as a real vector space of dimension $\dim_{\mathbb{R}} T_pR = 2 \dim_{\mathbb{C}} T_pR$ so the definition is consistent.

[5] The function differentiable k times

All of these ideas of sameness become incredibly important as we can represent both the space of all possible input values (Receptor space, R) and the abstract space of all output values (Scent Space, S) by smooth manifolds. The above notions of sameness allow us to relate the two spaces (relatively) by different continuous and differentiable mappings. This provides insight into the biological relation between the two spaces as homeomorphism admits an inverse. This general construction tells us that closed loops exist for olfaction. Specifically, feedback loops like those seen in standard biology fall out of the analysis fairly naturally as the inverse functions accompany the rest of the construction.

One advantage of this apporach allows for learning in the context of directional generalization. The model can learn to distinguish two stimuli which normally are generalized. All of this culminates to a theoretical solution for Blind Sources. Suppose you are placed in a room with no discernible features. An odor is then introduced, but it is impure. As a result, the brain must cope with such a situation by referencing what has already been learned. This problem, from a simulationist perspective, is hard: comparing an input to all similar(near) scents could rise exponentially as the number of specific receptor types increases. In our paradigm, we develop a method which relies on categorization to reduce the time it takes to indentify the input. This however, is completely dependent on the spaces we are dealing with and not the points themselves. In total, the model gives us a generalizable framework for which we can unify olfactor representations and learning whilst the latter transoforms the former.

R-Space and M-Space

R-Space

Now that we have the basic mathematical structures defined, we can proceed to construct the space of input or receptor/glomerular values, called $R-Space$ and denoted R. We know from biology that the number of distinct kinds of olfactory receptors is equal to the number of glomeruli. Suppose, for any given organism, that it has n glomeruli/receptor types. Now, suppose some scent, s^* is introduced to the receptor sites thus activiting the glomeruli. We can represent the activation in any glomerulus, g_i, as a number $x_i \in (a_i, b_i) \subseteq \mathbb{R}$. Now, notice that this scent s^* induces a response in each glomerulus so we can represent it by a point in the manifold constructed by the cartesian product of all glomeruli. We can think of this as each glomerulus defining a direction or degree of freedom which should be interpretted as the construction of a new dimension.

Definition 3.1. Let $g_i = (a_i, b_i)$ then $P = \prod_{1 \le i \le n} g_i$ is the space of all possible activation values.

Notice that viewing each glomerulus as an interval allows us to shrink and strech each glomerulus so that they become $g_i^* = (0, 1), 1 \le i \le n$.

Lemma 3.2 (Lemma/Definition). $P \cong \prod_{1 \le i \le n} g_i^* = \prod_{1 \le i \le n}(0, 1)$. *This is the glomerular/receptor space which we will denote, R.*

The proof of this lemma is presented in Appendix A.

Returning to the stimulus s^*, we can now represent its image as a point in this open unit $n-$cube, R. Due to the environment for which this odor is percieved and from natural variation, we know that s^* induces a variance Σ^2 which manifests as a structure analagous to a cloud in R. Essentially, it is uncertainty in the stimulus. Specifically, a stimulus, s^* has image a connected open submanifold of R. We will now prove that R itself is a smooth manifold. To do so, we introduce the idea of charts and atlases.

Definition 3.3. Let M be a manifold and (U, ϕ) a pair consisting of an open set U and a bijective map $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$. We call this pair a chart on M. Two charts (U, φ) and (V, ψ) are called compatible if

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is continuous or $U \cap V = \varnothing$. If $\psi \circ \varphi^{-1}$ is smooth, then they are said to be smoothly compatible.

Definition 3.4. An (smooth) atlas $\mathscr{A}$ is a collection of charts on M such that

1. $\bigcup_{U \in \mathscr{A}} U = M$
2. $\psi \circ \varphi^{-1}$ are (smoothly) compatible for all ψ, φ.

We can now formalize the definition of a smooth manifold. A manifold endowed with a maximal smooth atlas, denoted $(M, \mathscr{A})$ is called smooth. A simple example of this is $(\mathbb{R}^n, \{id_{\mathbb{R}^n}\})$.

Theorem 3.5. *R is a smooth manifold of dimension n. Specifically, R is an open submanifold of $\mathbb{R}^n$.*

Proof. As $\mathbb{R}$ is a product of intervals, it suffices to show that $(0,1)$ is an open submanifold of $\mathbb{R}$. Put, $\mathscr{A} = \{id_{(0,1)}\}$. This makes $(0,1)$ a smooth manifold. Therefore, R is a smooth manifold. To compute the dimension of R, notice that $\dim(0,1) = 1$. Therefore,

$$\dim R = \sum^{n} \dim(0,1) = n$$

Hence, R is a smooth manifold of dimension n. As a subset of $\mathbb{R}^n$ it is an open submanifold as it is open in the usual topology on $\mathbb{R}$. This completes the proof. $\square$

M-Space structure and actions

For the rest of this paper, we assume, for simplicity, that mitral cells are uniform within the glomeruli. That is to say, for each glomerulus we have one mitral cell. Therefore, let n be as above, then we have n mitral cells.

Mitral cells act similar to a vector space. To fit this structure together with our manifolds, we must first discuss tangent vectors. Let $x \in R$ and $c = (U, \varphi)$ a chart at a on R. Let $h \in \mathbb{R}^n$. Consider all triples (c, x, h). We define two triples, $(c_1, x_1, h_1), (c_2, x_2, h_2)$ to be equivalent, if they have the same base point and

$$h_2 = D(\varphi_2 \circ \varphi_1^{-1})(\varphi_1(x))h_1$$

where $D(f)$ is the total derivative or Jacobian of the funciton f.

Definition 3.6. A tangent vector is an equivalence class $[c, x, h]$. [6]

Definition 3.7. The collection of all possible tangent vectors to a (smoooth) manifold M is called the tangent bundle on M, and is denoted TM.

We can view TM in many different ways. The most convenient for this paper is that

$$TM = \coprod_{x \in M} T_x M = \{(x, v) | x \in M, v \in T_x M\}$$

Now consider R from above. We construct the tangent bundle TR. This space we have constructed gives a extremely tight model of mitral cells. The tangent bundle acts a highway to other mathematical objects as it not only encompases the manifold structure but it combines this with the vector space-like property we need.

[6] Notice that this makes $T_x M$ the collection of all tangent vectors based at the point x. This definition is equivalent to the definition given in the introduction.

Lemma 3.8. *Let U be an open submanifold of $\mathbb{R}^n$. Then $TU \overset{d}{\cong} U \times \mathbb{R}^n$.*[7]

Therefore, as R is an open submanifold of $\mathbb{R}^n$ we can apply Lemma 2.5 to see that TR is identified with $R \times \mathbb{R}^n$. Define mitral cells space or M space to be $M := TR$.

The reason this gives us an accurate model of mitral cells is because we can view a point in M as "remembering where it came from." What we mean by this is that for any point $(x, v) \in M$, we have a point from the manifold as well as a direction vector. As $x \in R$, for any stimulus, we get a cooresponding subset of the tangent bundle which coorsponds to all possible mitral outputs given that input.

So, at this point we have an incomplete diagram of the framework.

$$\begin{tikzcd} R \arrow[r, "\iota"'] & R \times \mathbb{R}^n = TR \arrow[l, bend right, "\pi_R"'] \end{tikzcd}$$

Now, there is another extremely important cell in mid-bulb which works in tendem with the mitral cells. These granule cells modify, amplify, and hinder the responses of the mitral cells to the input of the glomeruli. Essentially the granule cells act on their neighbouring mitral cells. If we think of granule cells as forming a group, then this action is quite interesting.

Definition 3.9. Let G be a group and S a set. A left action of G on S denoted, $G \circlearrowright S$ is a map $A : G \times S \to S$ such that $A(e, s) = s$ and $A(g, A(h, s)) = A(gh, s)$.

When we act on the tangent bundle, we have a choice to make. We can act on the vector part of an element, the manifold part of the element, or the entire element. If we only act on the vector portion of the bundle, then we are essentially acting on vector fields on R. Consider granule cells as group elements. Denote the group of granule cells by GC. Then we can add to the diagram above by including GC acting on TR.

$$\begin{tikzcd} R \arrow[r, "\iota"'] & TR \circlearrowleft GC \arrow[l, bend right, "\pi_R"'] \end{tikzcd}$$

We also know that mitral cells influence the granule cells in a way which is a feedback loop. This implies that we also need a map from $TR \to GC$. Normally, we do not consider a map from a set back into the group acting on

[7] Proof of this Lemma is omitted

it. For this to be defined, we need TR to look like $G \times R$ so that G forms the fibre of the action over TR. We know that

$$TR = R \times \mathbb{R}^n$$

Only because, R is an open submanifold of $\mathbb{R}^n$ can we then take the action of $\mathbb{R}^n$ on TR. However, we can canonically identify, $\mathbb{R}^n$ with the subgroup of $GL_n(\mathbb{R})$ consisting of diagonal matrices. Therefore, we have a lie group action G on TR and a map $TR \to \mathbb{R}^n$ by projection on the second element. This map is surjective but not injective. We conclude this section by adding this information to the diagram.

$$\begin{tikzcd} R \arrow[r, "\iota"'] & TR \arrow[l, bend right, "\pi_R"'] \arrow[r, bend right, "\pi_2"'] & GC \arrow[l, bend right, "Action"'] \end{tikzcd}$$

S-Space

Now that we have the base spaces, R and M, as well as the action GC, we can construct our space of interest. Consider the injection $R \hookrightarrow \mathbb{R}^{n+1}$. Specifically, consider R embedded in $\mathbb{R}^{n+1}$ shifted so that its center coincided with the origin. Let us denote this space also as R. Consider now, the "clouds" from before. Each cloud will be mapped to an open set in R canonically. As R has codimension 1 as a submanifold of $\mathbb{R}^{n+1}$, we can think of it as "flat" in $\mathbb{R}^{n+1}$ ($x_{n+1} = 0$). Therefore, each open set defined by a cloud can be "pulled up" into the ambient space. This pulling, if done carefully, creates the analogue to an $n-$dimensional distribution. Call this space, S.

Define the map $\Delta : R \to S$ as the spiking(pull out) of R. This map is a diffeomorphism trivially. In a system which has not had much learning, we have that these spikes, or peaks as we will refer to them, will possibly overlap. The discrimination between two similar stimuli will be discussed later. The properties of S space are more convenient to deal with than directly in R.

Metrics: An Introduction

Metrics are essential to any exploration of sensory perception as they are the key to defining the relations amongst elements in a manifold.

Definition 4.1. Let $x, y \in S$ be two points. We define the Physical Metric between the two points, as the Euclidean distance between them in R. In notation,

$$d_{phys}(x, y) = |\pi_{\mathbb{R}^n}(x) - \pi_{\mathbb{R}^n}(y)|$$

This metric reflects the physical similarities of the objects in the receptor space. We will return to this metric later when we discuss learning. For now, we will define another metric.

Definition 4.2. Let $x, y \in S$. Consider x and y as points. Then let P be the 2−dimensional affine plane defined by the vectors $[\pi_{\mathbb{R}^n}(y), y]$ and $[\pi_{\mathbb{R}^n}(y), x]$. Then, let $g : [0, 1] \to S$ be the curve defined by $S \cap P$ between x and y. The Perceptual Metric,

$$d^{per}(x, y) = \int_0^1 ||g'(t)||dt$$

This metric is incredibly important in the construction of learning. This metric accounts for the changes in S which are lost in the other metric. Suppose we grow one distribution (peak) further upward. d^{per} will tell us that the distances have increased between two points while d_{phys} will not tell us anything has changed. The main function of the two metrics is to track the changes of S as learning occurs.

Learning, Growing, and Discrimination

Let us revisit s^*. Suppose now, we have multiple instances of s^* in R. That is, we reintroduce the same stimulus many times. We observe in bulb that under these conditions, identification of a certain stumulus becomes more specific. Therefore, we know that the variance, Σ^2 of s^* will shrink with each trial. Formally, let $U_0 \subseteq R$ be the open subset Σ^2. The specification of learning tells us that at the second exposure, we have a contraction of $U_0 \to U_1$ such that

$$U_0 \supseteq U_1$$

If we have k instances, we get a chain

$$U_0 \supseteq U_1 \supseteq U_2 \supseteq ... \supseteq U_k$$

In theory, this sequence of U_i must converge, and it will converge onto s^* in R. We know this because U_i is a monotone decreasing sequence which is bounded below. In practice however, we see that $U_k \neq \{p\}$ for any k. Thus, there will always be variance even in a fully learned system. This is important for the sake of this model as some U_k being a point would break the continuity of the space.

Definition 4.3. The sequence $\{U_i\}_{i \leq k}$ of connected open sets is called the Secondary Learning Sequence of a stimulus.

For any stimulus, we get a corresponding secondary learning sequence. This is equivalent to saying that learning a stimulus makes the identification more accurate. Accompanying this sequence is the Primary Learning Sequence.

Definition 4.4. For any secondary learning sequence, we associate maps, $\Delta^i : U_i \to S$ such that Δ^i grows the peaks defined in S upwards further. The sequence $\{\Delta^i\}$ is called the Primary Learning Sequence.

The motivation for this definitions comes from the metrics imposed on S. The primary sequence is defined such that it preserves d_{phys} and changes only d^{per}. We can interpret these properties as: learning does not change the physical rleationships between stimuli but does change their perceptual relation. As we grow upwards, we see that the distance between two points along the surface of S will increase whilst the angle between them will not. Additionally, we are using the Δ^i to construct an attractor. As we learn the system better, we can discern more closely related stimuli.

Corollary 4.5. No static model (i.e. a (complicated) fixed tree diagram) will accurately predict perception, preserve physical data, and learning simultatneously.

Proof. Suppose for a contradiction, that a fixed model will suffice. Then, apply a single round of leanring and growth (take U_{i+1} and Δ^{i+1}) and the model will break as those values are not present. $\square$

Discrimination

We now will disucss discriminating between overlapping images in S. Consider the situation where we have that two peaks which overlap. During the process of learning, these two will become separated as the respective U_i, V_i become smaller. However, it is the case that we need disrimination between two like stimuli. Suppose we have s^* and t^* where $s^*, t^* \in U_k$ but $s^* \neq t^*$. We need to split these apart. The most convenient way to do this is with valleys or depressions.

Definition 4.6. Let s^* and t^* be two stimuli which define the same open neighbourhood, U_k. Let $[s^*, t^*]$ denote the shortest path along the surface of S between them. Let m be its midpoint. Define, $\Delta_i^\rho : [s^*, t^*] \to S$ which takes the path and pulls the midpoints down ($\rho x_{n+1} < x_{n+1}$) by a factor ρ.

This map acts to lengthen the perceptual metric d^{per} between two similar points, all the while keeping their physical separation equal.

So now we have, R, M, GC, S. We can now complete the main diagram from above.

$$
\begin{tikzcd}
R \arrow[r, "\iota"'] \arrow[d, "\Delta"'] & TR \arrow[l, bend right, "\pi_R"'] \arrow[r, bend right, "\pi_2"'] & GC \arrow[l, bend right, "Action"'] \\
S \arrow[ur] & &
\end{tikzcd}
$$

Note that although this diagram looks ststic, it is actually constantly being modified.

Search and Identification

We take a quick aside to discuss quotients and quotient spaces. Let X be a set.

Definition 5.1. We define a relation $\sim$ on X, such that the following properties hold

1. $\forall x \in X, x \sim x$
2. $a \sim b \iff b \sim a$
3. $a \sim b$ and $b \sim c \implies a \sim c$

These three conditions are reflexivity, symmetry, and transitivity respectively.

A trivial example on the real numbers is $a \sim b \iff a = b$. Notice that $\sim$ will partition X into sets where every element is equal to every other element in the set.

Definition 5.2. Let X be as above and $\sim$ an equivalence relation. Define

$$[x] = \{y \in X | y \sim x\}$$

This is called the equivalence class of X.

Lemma 5.3. *Let X be a set and $\sim$ an equivalence relation. Let $x, y \in X$. Then either $[x] = [y]$ or $[x] \cap [y] = \varnothing$.*

Proof. See Appendix A. $\square$

We can look at how X gets split by the relation by taking the space of equivalence classes, denoted

$$X/\sim$$

This is the quotient space of X under $\sim$ and can be thought of as what is left over after glueing the equivalent elements together. An interesting example of this is $\mathbb{R}$ under the equivalence relation $a \sim b \iff a - b \in \mathbb{Z}$. Under this relation, we get that

$$\mathbb{R}/\sim = \mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$$

This relation compactifies $\mathbb{R}$ to the interval $[0, 1]$ and then glues 0 to 1.

We need the notion of a quotient space to discuss a method for identification in S. We first view each peak as a continuous categorization for that stimulus (This is the image of a fully learned system). For instance we may have a peak defined for "oranges". As we move up the peak we refine the categorization. Here refinement means entering a subcategory. From the discussion above we know that the peak will be parsed into a variety of sub-peaks which correspond to physically similar but perceptually different types of orange. Pictured below is a complex of categories, ordered by inclusion

$$\text{Citrus Fruit } \supseteq \text{Oranges} \supseteq \text{Ripe Oranges } \supseteq \text{Ripe Valencia Oranges}$$

We can use the quotient operation to recover a coarse ordering for the categories in any given peak. Suppose P is a peak, determined by s^*, with several subpeaks $\{p_i\}_{i \in I}$. Then, as each sub-peak has a boundary, we can define the minimum value of p_i to be the point in p_i with the smallest x_{n+1} coordinate (denoted $\min p_i$), put

$$m_1 = \min\{\min p_i | i \in I\} = \min p_{i_1}, \text{ and } m_i = \min\{\min p_i | i \in I \setminus \{i_j\}_{j \leq i}\}$$

Further, let $m_0 = 0$. Then, for all points in P with x_{n+1} coordinate greater than m_0, we set $\sim_0$ to be the relation which identifies these elements as equivalent. In general, this will not work as the peak P may subdivide as we traverse upwards. So, let V be another copy of $\mathbb{R}^n$ not containing the origin. We will use V to divide the peak cleanly.

To do this, let V be such that it is parallel to S and $x_{n+1} < 0$ for all $x \in V$. We shall begin to increase this coordinate until V intersects S non-trivially. Suppose that $x_{n+1} = m_i$ for all $x \in V$. At this point, we compute the intersection, $V \cap S$. This gives us finitely many connected components within P. Now, let $\sim_i$ be the relation which identifies all points $x \in P$, such that x_{n+1} is greater than or equal to m_i and which are contained in the same connected component. So an equivalence class in $P/\sim_i$ looks like $[x]_{i,j}$ where j denotes inclusion to the $j-th$ connected component of the quotient. This gives us a discretization D_P of the smooth peak as well as a partial ordering $\preceq$ on D_P. That is, a categorization based on equivalence classes at each level as desired.

Discussion

Our method prioritizes the construction of a categorization out of an unordered space. To go the other direction (that is-to build a space out of a categorization) is more difficult and involves making choices along the way. This has been attempted by researchers and proven to be difficult. In a recent paper, by Sharpee et. al, they claim that there is a natural identification of the olfactory space with a hyperbolic geometric framework.[1] This follows from the choice of categorization of the space and the choice of representation as a tree.[2] To conclude from the data provided that the olfactory space is hyperbolic in structure does not follow from the data provided. As a corollary of what we have shown above, a fixed tree will not suffice to explain the dynamic nature of the space. We know this because for any fixed finite tree, the act of learning is not well defined as we need a non-isometric tranformation of the space as well as the ability to branch the diagram freely. For a fixed system, the choice of representing the system in hyperbolic space is perfectly acceptable but uninformative. Also, the model is dependent on the interpretation that there is a tree-like heirarchy in the perceptual space. Just because we think of the odors categorized locally in a simple tree, does not imply that they are globally organized in a tree. We also note the authors mention that their random data points, when plotted, approxiamted a sphere, $\mathbb{S}^2 = \partial\mathbb{H}^2$. This is to be expected as there is a natural identification of R with a half sphere. Additoinally, as presented in [2], the models of $\mathbb{H}^n$ and $\mathbb{S}^{n-1}$ are equivalent.

Further, the hyperbolic model does not take into consideration the biology of the system. It is wholly concerned with the "perceptual" piece of the "olfactory space." The main issue with this view is that we know perceptions dependends on the number of glomeruli and mitral cells. So to claim that the space depends solely on the categorization is false. A more accurate description of the olfactor space would need to include such considerations as presented above. Essentially, the paper attempted to investigate the opposite direction of what we have presented above, that is build a space out of a categorization and did provide sufficient evidence for such.

Conclusion

Future Projects

Appendix A: Proofs of Key Ideas

Proof of Lemma 2.2

Proof. It suffices to show that (a_i, b_i) is diffeomorphic to $(0,1)$ for a single $i \leq n$. Let $\lambda : (a_i, b_i) \to (0,1)$ be defined by

$$\lambda_i(x) = \frac{x - a_i}{b_i - a_i}$$

This function is bijective trivially. Additionally it is smooth and has inverse

$$\lambda^{-1}(y) = (b_i - a_i)y + a_i$$

which is also smooth. Hence they are diffeomorphic. This implies that $(a_i, b_i) \overset{d}{\cong} (0,1) \forall i \leq n$. So $\prod \lambda_i$ is a diffeomorphism and $P \overset{d}{\cong} R$. $\square$

Proof of Lemma 4.3

Proof. Suppose we have a set X and an equivalence relation $\sim$. Let $[x], [y]$ be two equivalence classes under the relation such that $[x] \cap [y] \neq \varnothing$. Then, as the intersection is non-empty, pick $z \in [x] \cap [y]$. As z must be in both equivalence classes, we know that $x \sim z$ and $y \sim z$. By the transitive axiom of an equivalence relation, this implies that $x \sim y$. Then, $[x] = [y]$. Hence,

$$[x] \cap [y] = [x] = [y]$$

For completion, if $[x] \cap [y] = \varnothing$ then the two sets are disjoint. This completes the proof. $\square$

Appendix B: Definitions

Definition 10.1. Let G be a set and $\mu : G \times G \to G$ be a binary operation on G. We call G a group if for all $a, b, c \in G$

1. $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$
2. $\exists e \in G$ with $\mu(e, a) = \mu(a, e) = a$
3. $\exists a^* \in G$ with $\mu(a, a^*) = \mu(a^*, a) = e$. We denote this element a^{-1}.

Definition 10.2. Let X, Y be topological spaces. A function $f : X \to Y$ is continuous if and only if for any $V \subset Y$ open, $f^{-1}(V)$ is open in X.

Definition 10.3. A diffeomorphism is is a function $f : X \to Y$ between smooth manifolds such that f is a smoth bijection with a smooth inverse.

Definition 10.4. A small category $\mathcal{C}$ is a pair $(Obj(\mathcal{C}), \mathrm{Hom}(\mathcal{C}))$ of objects and morphisms satisfying the following

1. If $A, B, C \in Obj(C)$ then $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set and there exists an associative map

$$M : \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

2. $id_A = 1_A \in \mathrm{Hom}(A, A)$

Definition 10.5. The small category C^∞ consists of the set of smooth manifolds, and $\mathrm{Hom}_F(C^\infty,)$ consisting of smooth maps. The isomorphism is this category is a diffeomorphism.

Appendix C: Notation

$\subseteq$, a subset. $A \subseteq B \implies a \in A$ then $a \in B$.
$A \to B$, denotes a mapping from the set A to the set B.
$f : A \to B$, a mapping from A to B by the function f.
$\circ$, composition
$\in$, element of
$\cup$, union $(A \cup B = \{c | c \in A \text{ or } B\})$
$\cap$, intersection $(A \cap B = \{c | c \in A \text{ and } B\})$
$\mathbb{C}$, the set of complex numbers.
$\mathbb{R}$, the set of real numbers.
$\mathbb{R}^n$, the set of all n-tuples of real numbers
$\mathbb{E}^n$ n-dimensional euclidean space (affine)
$\mathbb{Q}$, the set of rational numbers.
$\mathbb{Z}$, the set of integers.
$\mathbb{N} := \mathbb{Z}^+$, the set of natural numbers (non-negative integers)
$GL(n, \mathbb{R})$, the general linear group of all $n \times n$ invertible matrices.
$\Leftrightarrow$, if and only if.
$\det(A)$, the determinant of A.
$\prod$, product
$\sum$, sum
$\dim_{\mathbb{k}} A$, the dimension of a vector space or manifold over the field $\mathbb{k}$.

References

References

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